This homework is due by 11:59 PM on Tuesday, November 10.

Problem 1. Prove that if $\{x_n\}_{n=1}^{\infty}$ is an increasing sequence that is not bounded above, then $\lim_{n\to\infty} x_n = +\infty$.

Problem 2 (From Textbook problem 4.2.5)**.** Let $\{a_n\}_{n=1}^{\infty}$ be defined inductively as follows:

$$a_1 = 1, \qquad a_n = 1 + \frac{a_{n-1}}{4} \quad \text{for } n > 1$$

(a) Show by induction that a_n is bounded above by $4/3$.

(b) Show that $\{a_n\}_{n=1}^{\infty}$ is convergent by showing that it is increasing.

(c) Show that $\lim_{n\to\infty} a_n = 4/3$. [Hint: Use the fact that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$ and the recursive definition of a_n.]

Problem 3 (From Textbook problem 4.2.7)**.** Let $\{a_n\}_{n=1}^{\infty}$ be defined inductively as follows:

$$a_1 = 1, \qquad a_n = 1 + \frac{1}{1 + a_{n-1}} \quad \text{for } n > 1$$

(a) Show that $\{a_n\}_{n=1}^{\infty}$ converges by using the contraction principle. [Hint: First, show that $a_n \geq 1$ for all n.]

(b) Show that $\lim_{n\to\infty} a_n = \sqrt{2}$. [Hint: Use the fact that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$ and the recursive definition of a_n.]

Problem 4. Suppose that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are sequences, and $\{a_n\}_{n=1}^{\infty}$ is convergent with $\lim_{n\to\infty} a_n = L$. Suppose in addition that $\lim_{n\to\infty} |a_n - b_n| = 0$. Show that $\{b_n\}_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} b_n = L$.

Problem 5. Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(x) - f(y)| \leq r|x - y|$ for all $x, y \in \mathbb{R}$, where r is a constant in the interval $0 \leq r < 1$. Such a function is said to be a **contraction** on $\mathbb{R}$. Note that a contraction is simply a Lipschitz function with Lipschitz constant strictly less than 1, so we already know that f is continuous.

(a) Let t be any real number. Define a sequence $\{a_n\}_{n=0}^{\infty}$ by $a_0 = t$, $a_n = f(a_{n-1})$ for $n > 0$. That is $a_0 = t, a_1 = f(t), a_2 = f(f(t)), a_3 = f(f(f(t))), \ldots, a_n = f^n(t), \ldots$, where f^n is the composition of f with itself n times. Show that the sequence $\{a_n\}_{n=0}^{\infty}$ is contracting, and hence is convergent.

(b) Let $z = \lim_{n\to\infty} a_n$. Show that $f(z) = z$, that is, z is a fixed point of f. [Hint: Since f is continuous, $\lim_{n\to\infty} f(a_n) = f\big(\lim_{n\to\infty} a_n\big)$. Use this fact and the fact that $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n$.]

(Note: Recall that a **fixed point** of a function f is a point y such that $f(y) = y$. It is clear that a contraction can have at most one fixed point. This problem shows that a contraction always does have a fixed point. Furthermore, if t is any real number, then the sequence $\{f^n(t)\}_{n=0}^{\infty}$ converges to that unique fixed point. This is the **Contraction Mapping Theorem** for $\mathbb{R}$.)