This homework is due on Tuesday, November 17, by 11:59 PM. No rewrites are possible on this homework. Sample solutions for homeworks 8 and 9 will be published on the course web page on Wednesday, November 18. There is a test on Friday, November 20. A study guide for the test will be published on the course web page by Monday, November 16.

Problem 1. **(a)** Suppose that $\sum_{k=1}^{\infty} a_k$ is a convergent series, and $\sum_{k=1}^{\infty} b_k$ is a divergent series. Show that $\sum_{k=1}^{\infty}(a_k + b_k)$ diverges. [Hint: Proof by contradiction will work.]

(b) Suppose that $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are both divergent. Give a simple example to show that $\sum_{k=1}^{\infty}(a_k + b_k)$ does not necessarily diverge.

Problem 2. A nonnegative sequence must either converge (absolutely) or diverge to $+\infty$. Classify each of the following nonnegative series as either convergent or divergent. In all cases, explain your reasoning, being explicit about any convergence tests that you apply.

a) $\displaystyle\sum_{k=1}^{\infty} \frac{3k^2}{7k^5 + 2k^2}$ **b)** $\displaystyle\sum_{k=1}^{\infty} \frac{k}{\sqrt{k^2+1}}$ **c)** $\displaystyle\sum_{n=0}^{\infty} \pi^{-n}$ **d)** $\displaystyle\sum_{n=1}^{\infty} \frac{2^n + 3^n}{4^n}$

e) $\displaystyle\sum_{m=1}^{\infty} \frac{1+\sin(m)}{5^m}$ **f)** $\displaystyle\sum_{n=1}^{\infty} \frac{n^6}{5^n}$ **g)** $\displaystyle\sum_{k=1}^{\infty} \frac{(k!)^2}{(2k)!}$ **h)** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$

Problem 3. A series of positive and negative terms can either diverge, converge absolutely, or converge conditionally. Classify each of the following series as one of divergent, absolutely convergent, or conditionally convergent. In all cases, explain your reasoning, being explicit about any convergence tests that you apply.

a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{3^n}$ **b)** $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k+1}}$

c) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n \ln(n)}{n}$ **d)** $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k^k}$

Problem 4. The series $\sum_{k=2}^{\infty} \frac{1}{k\ln(k)}$ diverges, but none of the tests that we have covered can prove it. Note that $\lim\limits_{n\to\infty}\left(\int_2^n \frac{1}{x\ln(x)}dx\right) = \lim\limits_{n\to\infty}\big(\ln(\ln(x)) - \ln(\ln(2))\big) = +\infty$. Also note that $f(x) = \frac{1}{x\ln(x)}$ is decreasing. [You do not have to prove these facts.] Show that the partial sum, $s_n = \sum_{k=2}^{n} \frac{1}{k\ln(k)}$, satisfies

$$s_n \geq \int_2^{n+1} \frac{1}{x\ln(x)}\,dx$$

by considering the upper sum using the partition $\{2, 3, 4, \ldots, n+1\}$ of the interval $[2, n+1]$, and conclude that $\sum_{k=2}^{\infty} \frac{1}{k\ln(k)}$ diverges. (Note that this example is a special case of something called the "integral test.")