This test is due before class time on Monday, October 26. You should do this test on your own, using only your textbook, class notes, and previous work in the course as reference. You should not work with other students, you should not use the Internet or other references, and you should not consult anyone except me. You can ask questions about the test in class and by email. If I answer an emailed question, I will send my response to everyone in the class. I might give some hints and clarifications, but I am unlikely to give extensive, detailed help. Be sure to show all of your work! **Note: There will be no "rewrites" of midterm exam problems.** Grades on the exam might be curved, if they would lower the class average substantially.

This test counts for 15% of your overall grade for the course. **You should do only 7 of the 8 problems on this test.** If you submit answers for all problems, your answer for problem 8 will be ignored. If you cannot prove something, you should still turn in whatever work you can do on it for partial credit—maybe an intuitive argument or other thoughts. Although some problems are more difficult than others, and some are quite easy, all 7 problems will count equally.

Problem 1. Recall that $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + xy^{n-2} + y^{n-1})$ for any $n \in \mathbb{N}$. Use this fact to prove directly from the definition of the derivative that $\frac{d}{dx}x^n = nx^{n-1}$ for $n \in \mathbb{N}$.

Problem 2. Let f be an integrable function on $[a,b]$. Suppose that $A \le f(x) \le B$ for all $x \in [a,b]$. Show, from the definition of the integral, that $A \cdot (b-a) \le \int_a^b f \le B \cdot (b-a)$. (Hint: Use the trivial partition $P = \{x_0, x_1\}$ where $x_0 = a$, $x_1 = b$.)

Problem 3. Let A and B be subsets of $\mathbb{R}$. Suppose that x is an accumulation point of the set $A \cup B$. Show that x is an accumulation point of A or x is an accumulation point of B (or both). (Hint: Try a proof by contradiction.) If you prefer to work with subsets of a metric space (M,d) instead of $\mathbb{R}$, you can do that instead.

Problem 4. Suppose that $f(x)$ and $g(x)$ are uniformly continuous on the interval I (which is not necessarily closed or bounded). Show directly from the definition of uniform continuity that $f(x) + g(x)$ is uniformly continuous on I.

Problem 5. Let X and Y be non-empty, bounded subsets of $\mathbb{R}$. Suppose that for every $x \in X$ and for every $y \in Y$, $x < y$. Prove that $lub(X) \le glb(Y)$. Is it always true that $lub(X) < glb(Y)$?

Problem 6. This question concerns infinite sequences. You can write the proof for sequences in $\mathbb{R}$ or for sequences in a metric space. The proof would be essentially the same in either case; only the notation changes slightly.

Let $\{x_n\}_{n=1}^{\infty}$ be an infinite sequence. We define a *rearrangement* of the sequence as follows: Let $s\colon \mathbb{N} \to \mathbb{N}$ be a bijective function. Then $\{x_{s(i)}\}_{i=1}^{\infty}$ is a rearrangement of the sequence $\{x_n\}_{n=1}^{\infty}$. The rearranged sequence has exactly the same terms as the original sequence, just in a different order.

Suppose that $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence and that $\lim\limits_{n\to\infty} x_n = L$, and let $\{x_{s(i)}\}_{i=1}^{\infty}$ be a rearrangement of the sequence. Show that the rearranged sequence $\{x_{s(i)}\}_{i=1}^{\infty}$ is convergent and converges to the same limit, $\lim\limits_{i\to\infty} x_{s(i)} = L$. (Hint: This is easier than it looks. Note that the set $\{x_1, x_2, \ldots, x_N\}$ is finite. Also, s has an inverse function s^{-1}.)

Problem 7. Let (M, d) be a metric space. Recall that the open ball of radius ε about a point $x \in M$ is defined as $B_\varepsilon(x) = \{z \in M \mid d(z,x) < \varepsilon\}$. Suppose that we define the **closed ball** of radius ε about x as $\overline{B}_\varepsilon(x) = \{z \in M \mid d(z,x) \le \varepsilon\}$. We could also consider the closure of the open ball, $\overline{B_\varepsilon(x)}$, where the closure of a set consists of the set plus its accumulation points.

- **(a)** Show that $\overline{B}_\varepsilon(x) = \{z \in M \mid d(z,x) \le \varepsilon\}$ is a closed subset of M. (Hint: Show $M \smallsetminus \overline{B}_\varepsilon(x)$ is open, noting that $z \notin \overline{B}_\varepsilon(x)$ means $d(z,x) > \varepsilon$.)
- **(b)** In $\mathbb{R}$, $B_\varepsilon(x)$ is the open interval $(\varepsilon - \frac{\varepsilon}{2}, \varepsilon + \frac{\varepsilon}{2})$. What is $\overline{B}_\varepsilon(x)$ in $\mathbb{R}$? What is $\overline{B_\varepsilon(x)}$ in $\mathbb{R}$. You do **not** have to prove your answers, which should be obvious.
- **(c)** Give an example to show that $\overline{B}_\varepsilon(x)$ is **not** always equal to $\overline{B_\varepsilon(x)}$? (Hint: Consider a metric space with the discrete metric, and consider an open ball of radius 1.)

Problem 8. This problem proves that two integrable functions that differ at only one point have the same integral. (Note: A simple induction would then show that functions that agree except at a finite number of points have the same integral; **you do not need to prove this**.)

- **(a)** Let f be the function defined on $[a,b]$ by $f(x) = \begin{cases} 1 & \text{if } x = a \\ 0 & \text{if } x \ne a \end{cases}$. Show that f is integrable and $\int_a^b f = 0$. (Hint: For $\varepsilon > 0$, find a partition P of $[a,b]$ such that $L(P,f) \le 0 \le U(P,f)$ and $U(P,f) - L(P,f) < \varepsilon$. It is possible to do this with a specific partition that has just two subintervals.) A similar argument shows that a function whose value is 1 at b and 0 at other points of $[a,b]$ is integrable and has integral 0; **you do not need to prove this, but you can assume it for part (b)**.
- **(b)** Let $c \in (a,b)$ and consider the function $g(x) = \begin{cases} 1 & \text{if } x = c \\ 0 & \text{if } x \ne a \end{cases}$. Show that g is integrable and $\int_a^b g = 0$. (Hint: Use part (a) and the additivity of the integral.)
- **(c)** Let f be an integrable function on $[a,b]$. Suppose that g is a function that is defined on $[a,b]$, that $c \in [a,b]$, and that $g(x) = f(x)$ except that $g(c) \ne f(c)$. Show that g is integrable and $\int_a^b g = \int_a^b f$. (Hint: Use parts (a) and (b) and the linearity of the integral.)