## Math 331, Fall 2020, Test 1 Information

*The first in-class test will take place on Monday, October 5. It will cover Chapters 1 and 2 from the textbook and the section on open and closed sets from the metric spaces web site. You can expect questions that cover concepts, definitions, and theorems. You should certainly be able to give clear and precise statements of important definitions and theorems. You should be able to work with the concepts, definitions, and theorems as they apply to examples and problems. You should be able to do short proofs that are, from my point of view at least, simple and straightforward. The questions on the test will include shorter and longer essay-type questions, math problems (things like "find the least upper bound" or "find the following limit"), and simple proofs.*

*There will be **no** questions on the test about Dedekind cuts, the field axioms, limits at infinity, or infinite limits.*

**Some concepts, definitions, and theorems that you should know about for the test:**

the sets $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{N}$

rational and irrational numbers

examples of irrational numbers, such as $\sqrt[2]{p}$ for $p$ prime

bounded sets in $\mathbb{R}$; upper bounds and lower bounds

least upper bounds and greatest lower bounds

Archimedean property of $\mathbb{R}$

$\mathbb{R}$ is a complete, ordered field (and is characterized by this property)

a field is set with multiplication and addition satisfying certain axioms

an ordered field has an operation $<$ that is defined by a set, $P$, of positive elements

absolute value, $|x|$, and distance in $\mathbb{R}$, $|x-y|$

triangle inequality in $\mathbb{R}$: $|a+b| < |a|+|b|$, or $|x-z| < |x-y|+|y-z|$

open covers and subcovers

accumulation point of a set in $\mathbb{R}$

$\lim\limits_{x\to a} f(x)$, the epsilon-delta definition

properties of limits, such as the sum and product rules

continuity

uniform continuity

metric space; distance in a metric space; open ball $B_\varepsilon(x)$

open subset in a metric space; closed subset in a metric space

accumulation point of a set in a metric space

closure of a subset of a metric space

**Definition.** A subset $S$ of $\mathbb{R}$ is **dense** in $\mathbb{R}$ if for any $a, b \in \mathbb{R}$ with $a < b$, there exists an $s \in S$ such that $a < s < b$. (Alternative definition: $S$ is dense if for every non-empty open subset $U$ of $\mathbb{R}$, $S \cap U \neq \varnothing$.)

**Definition.** An **accumulation point** of a subset $X$ of $\mathbb{R}$ is point $a \in \mathbb{R}$ such that for any $\varepsilon > 0$, there is an $x \in X$ such that $0 < |x-a| < \varepsilon$.

**Definition.** An **open cover** for a subset $X$ of $\mathbb{R}$ is a collection of open sets $\{\mathcal{O}_\alpha \,|\, \alpha \in A\}$ such that $X \subseteq \bigcup_{\alpha \in A} \mathcal{O}_\alpha$. A **subcover** for this cover is a subset of $\{\mathcal{O}_\alpha \,|\, \alpha \in A\}$ that is still a cover of $X$.

**Definition.** For a function $f$ and $a, L \in \mathbb{R}$, we say $\lim_{x \to a} f(x) = L$ if for every $\varepsilon > 0$, there is a $\delta > 0$ such that for any $x$, $0 < |x - a| < \delta$ implies $|f(x) - L| < \varepsilon$.

**Definition.** For a function $f$ and $a \in \mathbb{R}$, we say $f$ is **continuous** at $a$ if for every $\varepsilon > 0$, there is a $\delta > 0$ such that for all $x$, if $|x - a| < \delta$ then $|f(x) - f(a)| < \varepsilon$. (Equivalently, $f$ is continuous at $a$ if $\lim_{x \to a} f(x) = f(a)$.)

**Definition.** A function $f$ is **uniformly continuous** on an interval $I$ if for every $\varepsilon > 0$, there is a $\delta > 0$ such that for all $x, y \in I$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

**Definition.** A **metric space** is a pair $(M, d)$, where $M$ is a set and $d\colon M \times M \to \mathbb{R}$, satisfying (1) $d(x, y) > 0$ for all $x, y \in M$; (2) $d(x, y) = 0$ if and only if $x = y$, for all $x, y \in M$; (3) $d(x, y) = d(y, x)$ for all $x, y \in M$; and (4) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in M$. (Property (4) is the **triangle inequality**.)

**Definition.** If $(M, d)$ is a metric space, $x \in M$, and $r > 0$, then the **open ball** about $x$ of radius $r$ in the metric $d$ is defined as $B^d_r(x) = \{z \in M \,|\, d(z, x) < r\}$.

**Definition.** A subset $U$ of a metric space $(M, d)$ is **open** if for every $x \in U$, there is an $\varepsilon > 0$ such that $B^d_\varepsilon(x) \subseteq U$. A subset $C$ is **closed** if $M \smallsetminus C$ is open.

**Definition.** Let $X$ be a subset of a metric space $M$. A point $z \in M$ is an **accumulation point** of $M$ if for all $\varepsilon > 0$, there is some $x \in X$ such that $0 < d(x, z) < \varepsilon$ (equivalently, $(B_\varepsilon(z) \smallsetminus \{z\}) \cap X \neq \varnothing$).

**Theorem.** (Completeness of $\mathbb{R}$.) Every non-empty subset of $\mathbb{R}$ that is bounded above has a least upper bound. (It follows that every non-empty subset of $\mathbb{R}$ that is bounded below has a greatest lower bound.)

**Theorem.** (Archimedean property of $\mathbb{R}$.) For any positive $x \in R$, there is an $n \in \mathbb{N}$ such that $n > x$. (Equivalently: For any positive $a, b \in \mathbb{R}$, there is an $n \in \mathbb{N}$ such that $an > b$.)

**Theorem.** (Heine-Borel Theorem.) Every open cover of a bounded closed interval in $\mathbb{R}$ has a finite subcover

**Theorem.** (Bolzano-Weirstrass Theorem.) Every bounded infinite subset of $\mathbb{R}$ has an accumulation point.

**Theorem.** (Intermediate Value Theorem.) If the function $f$ is continuous on the closed, bounded interval $[a, b]$, and $y$ is strictly between $f(a)$ and $f(b)$, then there is a $c \in (a, b)$ such that $f(c) = y$.

**Theorem.** (Extreme Value Theorem.) If the function $f$ is continuous on the closed, bounded interval $[a, b]$ then there exist $x_0, x_1 \in [a, b]$ such that for all $x \in [a, b]$, $f(x_0) \leq f(x) \leq f(x_1)$.

**Theorem.** If the function $f$ is continuous on the closed, bounded interval $[a, b]$, then it is uniformly continuous on $[a, b]$.

**Theorem.** (Properties of open sets.) Let $(M, d)$ be a metric space. Then (1) $\varnothing$ and $M$ are open subsets of $M$; (2) the union of any collection of open subsets of $M$ is open; and (3) the intersection of any finite collection of open subsets of $M$ is open.

**Theorem.** (Properties of closed sets.) Let $(M, d)$ be a metric space. Then (1) $\varnothing$ and $M$ are closed subsets of $M$; (2) the intersection of any collection of closed subsets of $M$ is closed; and (3) the union of any finite collection of closed subsets of $M$ is closed.