Outline of Complex Fourier Analysis

This handout is a summary of three types of Fourier analysis that use complex numbers: Complex Fourier Series, the Discrete Fourier Transform, and the (continuous) Fourier Transform.

1 Complex Fourier Series

The set of complex numbers is defined to be $\mathbb{C} = \{x + iy \,|\, x, y \in \mathbb{R}\}$, where $i = \sqrt{-1}$. The complex numbers can be identified with the plane, with $x + iy$ corresponding to the point (x, y). If $z = x + iy$ is a complex number, then x is called its *real part* and y is its *imaginary part*. The *complex conjugate* of z is defined to be $\overline{z} = x - iy$. And the length of z is given by $|z| = \sqrt{x^2 + y^2}$. Note that $|z|$ is the distance of z from 0. Also note that $|z|^2 = z\overline{z}$.

The complex exponential function $e^{i\theta}$ is defined to be equal to $\cos(\theta) + i\sin(\theta)$. Note that $e^{i\theta}$ is a point on the unit circle, $x^2 + y^2 = 1$. Any complex number z can be written in the form $re^{i\theta}$ where $r = |z|$ and θ is an appropriately chosen angle; this is sometimes called the *polar form* of the complex number. Considered as a function of θ, $e^{i\theta}$ is a periodic function, with period 2π. More generally, if $n \in \mathbb{Z}$, e^{inx} is a function of x that has period 2π.

Suppose that $f\colon \mathbb{R} \to \mathbb{C}$ is a periodic function with period 2π. The *Complex Fourier Series* of f is defined to be

$$\sum_{n=-\infty}^{\infty} c_n e^{inx}$$

where c_n is given by the integral

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx$$

for $n \in \mathbb{Z}$. The numbers c_n are called the complex Fourier coefficients of f. The Fourier series is only defined if all these integrals are defined. This will be true, for example, if f is continuous or if f is bounded and has only a finite number of jump discontinuities; however, weaker conditions will also work. You should think of the doubly infinite sequence $\{c_n\}_{n=-\infty}^{n=\infty}$ as being a transform of f, while the Fourier series is actually the inverse transform that takes the sequence $\{c_n\}_{n=-\infty}^{n=\infty}$ and uses it to define a function. In fact, if f is continuous, then the Fourier series will converge to f, and if f is continuous except for a finite number of jump discontinuities, then the series converges to f at every point of continuity while at the jump discontinuities it converges to the average of the left and right hand limits.

Complex vector spaces can be defined in exactly the same way as real vector spaces, except that scalar multiplication refers to multiplication by complex numbers rather than multiplication by real numbers. Linear independence and bases are also defined just as in the case of real vector spaces. The set $\mathbb{C}^n$ is a complex vector space of dimension n over $\mathbb{C}$. (It can also be considered a real vector space of dimension $2n$ over $\mathbb{R}$.) In $\mathbb{C}^n$, we define the *scalar product* of two vectors $\mathbf{z} = (z_1, z_2, \ldots, z_n)$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ to be

$$\mathbf{z} \cdot \mathbf{w} = z_1\overline{w_1} + z_2\overline{w_2} + \cdots + z_n\overline{w_n}$$

Note that $\mathbf{z} \cdot \mathbf{z} = z_1\overline{z_1} + z_2\overline{z_2} + \cdots + z_n\overline{z_n} = |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2$. In particular, $\mathbf{z} \cdot \mathbf{z}$ is a positive real number. We define the norm $|\mathbf{z}| = \sqrt{\mathbf{z} \cdot \mathbf{z}}$.

The set of continuous complex-valued functions $f \colon \mathbb{R} \to \mathbb{C}$ with period 2π is a complex vector space. We define a scalar product on this vector space by

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\overline{g(x)}\, dx$$

With this definition, we have

$$\langle f, f \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\overline{f(x)}\, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2\, dx$$

so $\langle f, f \rangle$ is a positive real number. We define the norm of f by $|f| = \sqrt{\langle f, f \rangle}$. This scalar product and norm have all the important properties of scalar product and length for finite dimensional vector spaces. Note that the same definitions will work on larger spaces of functions, such as functions that are continuous except for a finite number of jump discontinuities.

If we consider the periodic functions e^{inx}, for $n \in \mathbb{Z}$ with respect to this scalar product, we see that $\langle e^{inx}, e^{inx} \rangle = 1$ and $\langle e^{inx}, e^{imx} \rangle = 0$ for $n \neq m$. Thus, the set $\{e^{inx} \,|\, n \in \mathbb{Z}\}$ is an orthonormal set of vectors. In fact, it is a so-called "Hilbert basis" of a certain vector space of functions. (Note that a Hilbert basis is not the same as an ordinary basis; for an ordinary basis, every vector is a *finite* linear combination of basis vectors; for a Hilbert basis, *infinite* linear combinations are allowed, and there are issues of convergence of infinite series that require some advanced mathematics to deal with.) The complex Fourier coefficients of a function f are given by $c_n = \langle f, e^{inx} \rangle$, so that the c_n are just the components of f with respect to the Hilbert basis. The equation $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$ is just the usual statement about writing a vector in terms of an orthonormal basis (ignoring the issue of convergence).

(In case you are curious.... The correct vector space of functions to use for Fourier series is the space of *square-summable* functions, that is, those for which $\int_{-\pi}^{\pi} |f(x)|^2 dx < \infty$. However, the integral in this formula must be taken to be a *Lesbegue integral*, which is a generalization of the Riemann integral. The sequence of Fourier coefficients $\{c_n\}_{n=-\infty}^{n=\infty}$ of a function in this space satisfies $\sum_{n=-\infty}^{\infty} |c_n|^2 < \infty$. That is, the series of Fourier coefficients is also square-summable. In fact, $\sum_{n=-\infty}^{\infty} |c_n|^2 = \int_{-\pi}^{\pi} |f(x)|^2 dx$ and we get a length-preserving isomorphism from the Hilbert space of square-summable functions to the Hilbert space of square-summable doubly infinite sequences.)

Note, by the way, that for a periodic function f with period T, we can define the Fourier coefficients of f as $c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(x) e^{-2\pi inx/T} dx$ or, equivalently, $c_n = \frac{1}{T} \int_0^T f(x) e^{-2\pi inx/T} dx$. In particular, for a function that has period 1, we have $c_n = \int_0^1 f(x) e^{-2\pi inx} dx$.

2 Discrete Fourier Transform

Suppose that $\mathbf{x} = (x_0, x_2, \ldots, x_{N-1})$ is a finite, discrete, complex-valued signal, where $x_0, x_1, \ldots, x_{N-1}$ are complex numbers. (That is, $\mathbf{x}$ is a member of the complex vector space $\mathbb{C}^N$.) We define the Discrete Fourier Transform (DFT) of $\mathbf{x}$ to be the signal $\hat{\mathbf{x}} = (\hat{x}_0, \hat{x}_2, \ldots, \hat{x}_{N-1})$, where

$$\hat{x}_j = \sum_{k=0}^{N-1} x_k e^{-2\pi ijk/N}$$

The numbers $e^{-2\pi ijk/N}$ are all integral powers of $\omega = e^{2\pi i/N}$. ω is an N^{th} root of unity; that is, it satisfies $\omega^N = 1$. Every integral power of ω is also an N^{th} root of unity, and in fact the complete set of N^{th} roots of unity is given by $\{\omega^0, \omega^1, \ldots, \omega^{N-1}\}$. Using ω, we can write the definition of $\hat{x}$ as

$$\hat{x}_j = \sum_{k=0}^{N-1} x_k \omega^{-jk}$$

In fact, we can go further and define vectors $\mathbf{w}_0, \mathbf{w}_1, \ldots \mathbf{w}_{N-1}$ by

$$\mathbf{w}_j = (\omega^{0j}, \omega^{1j}, \omega^{2j}, \ldots, \omega^{(N-1)j})$$

so that we then have

$$\hat{x}_j = \mathbf{x} \cdot \mathbf{w}_j$$

for $j = 0, 1, \ldots, (N-1)$. Now, $\{\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_{N-1}\}$ is an orthogonal set of vectors in $\mathbb{C}^N$ since

$$\mathbf{w}_j \cdot \mathbf{w}_j \;=\; \sum_{j=0}^{N-1} |\omega^{kj}|^2 \;=\; \sum_{j=0}^{N-1} 1 \;=\; N$$

while for $k \neq n$

$$\mathbf{w}_j \cdot \mathbf{w}_n \;=\; \sum_{j=0}^{N-1} \omega^{kj}\overline{\omega^{nj}} \;=\; \sum_{j=0}^{N-1} \omega^{kj}\omega^{-nj} \;=\; \sum_{j=0}^{N-1} (\omega^{k-n})^j \;=\; \frac{1-(\omega^{k-n})^N}{1-\omega^{k-n}} \;=\; \frac{1-1}{1-\omega^{k-n}} = 0$$

(The next-to-last equality follows since $\sum_{j=0}^{N-1}(\omega^{k-n})^j$ is a geometric series with ratio ω^{k-n}.) Since it contains N mutually orthogonal vectors, the set $\{\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_{N-1}\}$ is actually a basis of $\mathbb{C}^N$. The components of a signal $\mathbf{x} = (x_0, x_2, \ldots, x_{N-1})$ with respect to this basis are given by $\frac{\mathbf{x}\cdot\mathbf{w}_j}{\mathbf{w}_j\cdot\mathbf{w}_j} = \frac{1}{N}\mathbf{x}\cdot\mathbf{w}_j = \frac{1}{N}\hat{x}_j$, for $j = 0, 1, \ldots, (N-1)$. Saying that these are the components of $\mathbf{x}$ means simply that

$$\mathbf{x} \;=\; \sum_{j=0}^{N-1}\left(\frac{1}{N}\hat{x}_j\right)\mathbf{w}_j \;=\; \frac{1}{N}\sum_{j=0}^{N-1}\hat{x}_j\mathbf{w}_j$$

or, equivalently, that for $k = 0, 1, \ldots, (N-1)$,

$$x_k \;=\; \frac{1}{N}\sum_{j=0}^{N-1}\hat{x}_j\omega^{kj} \;=\; \frac{1}{N}\sum_{j=0}^{N-1}\hat{x}_j e^{2\pi ijk/N}$$

This is the inverse discrete Fourier transform. (Note how neatly this follows when we think in terms of orthogonal bases!)

Let's look at the relationship between complex Fourier series and the DFT. Suppose that we start with a function $f\colon \mathbb{R} \to \mathbb{C}$ that has period 1. Given $N \in \mathbb{Z}^+$, we can obtain a signal of length N by sampling f at N evenly spaced points in the interval $[0, 1]$. Specifically, consider the signal $\mathbf{f} = \big(f(0), f(\frac{1}{N}), f(\frac{2}{N}), \ldots, f(\frac{N-1}{N})\big)$. The DFT of $\mathbf{f}$ is a signal $\hat{\mathbf{f}}$ of length N whose components are given by

$$\hat{f}_n = \sum_{k=0}^{N-1} f(x)e^{-2\pi ink/N}$$

while the Fourier coefficients of f are given by

$$c_n = \int_0^1 f(x)e^{-2\pi inx}\,dx$$

For $n = 0, 1, \ldots (N-1)$, we recognize $\hat{f}_n$ as being almost equal to a simple Riemann sum approximation for c_n, based on a partition of the interval $[0, 1]$ with division points k/N for $k = 0, 1, \ldots, (N-1)$. All that's missing is a factor of $\frac{1}{N}$, which represents the length of the subintervals: $c_n = \int_0^1 f(x)e^{-2\pi inx}\,dx \approx \sum_{k=0}^{N-1} f(x)e^{-2\pi ink/N}\frac{1}{N} = \frac{1}{N}\hat{f}_n$. Furthermore, as $N \to \infty$, the formula for $\frac{1}{N}\hat{f}_n$ will converge to the formula for c_n (provided that f is Riemann integrable). So, from the point of view of sampling, the DFT is a kind of approximation of complex Fourier series that improves as the number of samples is increased. Numerical applications generally use the DFT, rather than using Fourier series directly, but the information that is obtained is essentially the same.

The DFT is a linear transformation from $\mathbb{C}^N$ to itself. The matrix, D, of this linear transformation has ω^{jk} as the entry in row j, column k, for $j, k = 0, 1, \ldots (N-1)$. That is (using the fact that $\omega^0 = 1$),

$$\begin{pmatrix} \hat{x}_0 \\ \hat{x}_1 \\ \hat{x}_2 \\ \vdots \\ \hat{x}_{N-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^1 & \omega^2 & \cdots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{N-1} & \omega^{(N-1)2} & \cdots & \omega^{(N-1)(N-1)} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{N-1} \end{pmatrix}$$

Evaluating the matrix product in a straightforward way would require N^2 multiplications and about the same number of additions. For large values of N, this would be very time consuming, even on a fast computer. However, when N is a power of 2, the matrix D can be factored in a clever way into a product of simpler matrices in a way that will reduce the total number of operations to something on the order of $N\log(N)$, which is much smaller than N^2. The DFT when computed in this way is called the *Fast Fourier Transform* (FFT). The discovery of the FFT made it possible to compute the DFT quickly and efficiently. In fact, it is fast enough to be applied to signal analysis in real time and it is the key innovation that has made real-time digital signal processing practical. (We might return to the details of the FFT later.)

3 The Fourier Transform

The Fourier Transform is defined for (non-periodic) functions $f\colon \mathbb{R} \to \mathbb{C}$. Not every function has a Fourier Transform. A function that is integrable ($\int_{-\infty}^{\infty} |f(x)|dx < \infty$) will have a Fourier transform, but looser restrictions are possible. The Fourier transform of a function $f(x)$ is defined to be the function $\hat{f}(\xi)$ given by

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}dx$$

The "test function" $e^{2\pi i x\xi}$, as a function of x, is periodic with period $2\pi/\xi$ (when $\xi \neq 0$), or equivalently with frequency $\xi/(2\pi)$, so integrating f against this test function gives information about the extent to which this frequency occurs in f. However, things are not so simple as in the Fourier series case, since the functions $e^{2\pi i x\xi}$ for $\xi \in \mathbb{R}$ do not form an orthonormal set in any sense. In fact, these functions are not even integrable. Nevertheless, we do get an inversion formula

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi i x\xi}d\xi$$

This formula will hold for all x if f is a continuous and integrable and if $\hat{f}(\xi)$ is also integrable. In other cases, it is more problematic.

For now, lets just look at how the Fourier transform relates to complex Fourier series and to the DFT. The integral $\int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}dx$ can be written as a limit of integrals over finite intervals $[-T, T]$:

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}dx = \lim_{T\to\infty} \int_{-T/2}^{T/2} f(x)e^{-2\pi i x\xi}dx$$

For a fixed T, we can consider f restricted to the interval $[-T/2, T/2)$ and find its Fourier coefficients

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(x)e^{-2\pi i x n/T}dx$$

for all $n \in \mathbb{Z}$. Comparing this with the above limit, we see that $c_n \approx \frac{1}{T}\hat{f}\left(\frac{n}{T}\right)$. Note that the larger the value of T, the better the approximation and the more closely spaced the "samples" $\hat{f}\left(\frac{n}{T}\right)$. We already know that Fourier series coefficients can be approximated using the DFT. Since a Fourier transform can be approximated using Fourier series coefficients, we see that Fourier transforms can be approximated in turn using the DFT.

We can also compare the Fourier inversion formula to Fourier series. Using the Fourier coefficients for f restricted to the interval $[T/2, T/2)$, we get $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i x n/T} \approx \sum_{n=-\infty}^{\infty} \left(\frac{1}{T}\hat{f}(\frac{n}{T})\right)e^{2\pi i x n/T} = \sum_{n=-\infty}^{\infty} \hat{f}(\frac{n}{T})e^{2\pi i x n/T}\frac{1}{T} \approx \int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi i x\xi}d\xi$, where the last equality follows because the previous sum is a Riemann sum approximation for the integral, using subintervals of length $\frac{1}{T}$. Thus, we have at least $f(x) \approx \int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi i x\xi}d\xi$, and we can expect the approximation to improve as $T \to \infty$, giving the exact inversion formula.

Let's look briefly at some of the properties of the Fourier transform. An alternative notation for the Fourier transform $\hat{f}$ of a function f is $\mathcal{F}(f)$, or sometimes simply $\mathcal{F}f$. This notation will make it easier to write some of the properties.

First of all, of course, the Fourier transform is a linear transformation. For any functions f and g that have Fourier transorms and any $\alpha, \beta \in \mathbb{C}$, the function $\alpha f + \beta g$ has a Fourier transform, and $\mathcal{F}(\alpha f + \beta g) = \alpha\mathcal{F}(f) + \beta\mathcal{F}(g)$. This follows easily from the linearity of the definite integral.

Suppose we apply a horizontal translation to a function. That is, let $g(x) = f(x - x_o)$, where $x_o \in \mathbb{R}$ is a constant. Using a substitution $u = x - x_o$ in the integral, we get

$$\hat{g}(\xi) = \int_{-\infty}^{\infty} f(x - x_o)e^{-2\pi ix\xi}dx = \int_{-\infty}^{\infty} f(u)e^{-2\pi i(u+x_o)\xi}du$$

$$= e^{-2\pi ix_o\xi}\int_{-\infty}^{\infty} f(u)e^{-2\pi iu\xi}du = e^{-2\pi ix_o\xi}\hat{f}(\xi)$$

We see that applying a horizontal translation to $f(x)$ will multiply its fourier transform $\hat{f}(\xi)$ by the function $e^{-2\pi ix_o\xi}$ (a function of ξ). Note that this function has constant norm 1, so multiplying $\hat{f}(\xi)$ does not change the norm $|\hat{f}(\xi)|$; it can be thought of as a change in *phase* in the transform. The norm $|\hat{f}(\xi)|$ is called the *spectrum* or *power spectrum* of f. We see that f has the same power spectrum as any of its horizontal translates. In other words, the power spectrum tells us something about the magnitude of each frequency component in f, but tells us nothing about *where* that frequency component occurs—that information is hidden in the phase.

For a continuous function to have a Fourier transform, its limits at ∞ and $-\infty$ must be zero, since otherwise the integral from ∞ to $-\infty$ would not be finite. There is an interesting relationship between *how fast* $f(x)$ approaches 0 at $\pm\infty$ and *how smooth* its Fourier transform is. The converse relationship (between smoothness of $f(x)$ and speed with which $\hat{f}(\xi)$ converges to 0) also holds. Here, "smooth" means being differentiable, and being more smooth means having derivatives of higher order. We can quantify the speed at which a function $f(x)$ approachs 0 by looking at the behavior of $x^n f(x)$ for positive integers n. If $\lim\limits_{x\to\pm\infty} xf(x) = 0$, then f must approach 0 rapidly enough to balance the fact that x is approaching ∞; if $\lim\limits_{x\to\pm\infty} x^2 f(x) = 0$, then f must approach 0 even more rapidly; and so on. The relationship between convergence on one side of the Fourier transform and smoothness on the other is a consequence of the following theorems:

Theorem: Let $f\colon \mathbb{R} \to \mathbb{C}$ be a function, and let $g(x) = xf(x)$. Suppose that both f and g are integrable on $\mathbb{R}$. Then the Fourier transform of f is a differentiable function, and in fact $\hat{g}(\xi) = \frac{-1}{2\pi i}\hat{f}\,'(\xi)$.

Theorem: Let $f\colon \mathbb{R} \to \mathbb{C}$ be a function, and suppose that $f(x)$, $\hat{f}(\xi)$, and $\xi\hat{f}(\xi)$ are all integrable on $\mathbb{R}$. Then f is differentiable and

$$f'(x) = 2\pi i\int_{-\infty}^{\infty} \xi\hat{f}(\xi)e^{2\pi ix\xi}d\xi$$