Math 371, Spring 2006 — Homework #1

*Due **in class** on Friday, January 27*

Part 1: Haar Wavelets

1. Find the one-level Haar transform of the finite discrete signal $\mathbf{f} = (1, 0, 2, 4, 7, 10, 12, 13, 11, 9, 7, 5)$.

2. Consider a *constant* signal $\mathbf{f} = (c, c, c, \ldots, c)$ of length 2^n, and consider the full n-level Haar transform $H^n(\mathbf{f}) = (\mathbf{a}^n \,|\, \mathbf{d}^n \,|\, \mathbf{d}^{n-1} \,|\, \ldots \,|\, \mathbf{d}^1)$. Compute $H_n(\mathbf{f})$. Justify your answer.

3. Find a signal $\mathbf{f}$ such that the one-level Haar transform of $\mathbf{f}$ is $H_1(\mathbf{f}) = (0, 1, -1, 0, 6, 2, 7, -5)$. Does this example exhibit *compaction of energy* as discussed on page 7 of the textbook? Write a paragraph discussing the idea of compaction of energy, the circumstances under which it applies, and why the idea might be useful, even though it doesn't apply in all cases. (For more credit, consider the question of what happens when you look at a "random signal".)

Part 2: Sines, Cosines, and Fourier Series

*We looked briefly at the idea of writing a periodic function as a Fourier series, that is as a sum of sine and cosine functions. This part of the homework explores the idea further. To keep things simple, assume that the period is 2π. ($L = \pi$ in the formulas from class.) In this case, a Fourier series has the form $f(x) = a_o + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$. The first question is, why do we only use sine and cosine functions in which the argument is an **integral** multiple of x? The answer is that we want functions that have period 2π. A function g has period 2π if $g(x + 2\pi) = g(x)$ for all x.*

4. Show that the function $g(x) = \sin(rx)$, where r is a positive real number, has period 2π if and only if r is a positive integer. (The same is true for cosine functions; you do not have to prove this.)

The wavelength of a periodic function is the length of its shortest period. (Frequency is just the reciprocal of wavelength.) For example, $\sin(nx)$, where $n \in \mathbb{Z}^+$, has period 2π, but it is also periodic with period $2\pi/n$, and $2\pi/n$ is the smallest number with this property. So the wavelength of $\sin(nx)$ is $2\pi/n$. The Fourier series, therefore, is a constant plus a sum of functions of wavelengths 2π, $2\pi/2$, $2\pi/3$, $2\pi/4$, $\ldots$. (These numbers are fractions 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\ldots$, of the "fundamental" period 2π.) However, note that there are two functions in the sum with wavelength $2\pi/n$, namely $a_n \cos(nx)$ and $b_n \sin(nx)$. Why the redundancy? If we are looking for "the" component of $f(x)$ that has a given wavelength, why does it come in two pieces?

5. Use the standard trigonometric identity $\sin(s + t) = \sin(s)\cos(t) + \cos(s)\sin(t)$ to show that $a_n \cos(nx) + b_n \sin(nx)$ can be written in the form $c \sin(nx + d)$, for some real numbers c and d with $c \geq 0$ and $-\pi \leq d < \pi$. (If a and b are not both zero, then $c > 0$.) Find explicit formulas for c and d in terms of a_n and b_n. (Hint: apply the formula to $c \sin(nx + d)$.)

We can rewrite $c \sin(nx + d)$ as $c_n \sin(n(x - \varphi_n))$, where $c_n = c$ and $\varphi_n = -d/n$. Note that we then have $-\pi/n < \varphi_n \leq \pi/n$. We can then write the Fourier series as $a_o + \sum_{n=1}^{\infty} c_n \sin(n(x - \varphi_n))$. Then, c_n tells us the size of the wavelength-$2\pi/n$ component of $f(x)$. What about φ_n? (φ_n is called the "phase" of the wave.)

6. Write a paragraph discussing the effect that the phase, φ_n, has on the wave $c_n \sin(n(x-\varphi_n))$. What happens as φ_n varies from $-\pi$ to π? Draw some pictures.

Finally, let's return to the question of finding the constants a_n and b_n. For functions with period 2π, the formulas become $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cos(nx)\,dx$ and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$. The proof of this uses the following formulas, which hold for all integers n and k:

a) $\displaystyle\int_{-\pi}^{\pi} \sin(nx)\,dx = 0$

b) $\displaystyle\int_{-\pi}^{\pi} \cos(nx)\,dx = 0$

c) $\displaystyle\int_{-\pi}^{\pi} \sin(kx)\cos(nx)\,dx = 0$

d) $\displaystyle\int_{-\pi}^{\pi} \sin(kx)\sin(nx)\,dx = 0$, provided $n \neq k$

e) $\displaystyle\int_{-\pi}^{\pi} \sin(kx)\sin(nx)\,dx = \pi$, if $n = k$

f) $\displaystyle\int_{-\pi}^{\pi} \cos(kx)\cos(nx)\,dx = 0$, provided $n \neq k$

g) $\displaystyle\int_{-\pi}^{\pi} \cos(kx)\cos(nx)\,dx = \pi$, if $n = k$

*Formulas **a)** and **b)** are trivial, using the fact that $\sin(nx)$ and $\cos(nx)$ have period 2π. The other formulas can be proved in various ways. Perhaps the easiest proofs use the following identities for the product of sine and cosine functions:*

g) $\sin(s)\sin(t) = \frac{1}{2}\big(\cos(s-t) - \cos(s+t)\big)$

h) $\cos(s)\cos(t) = \frac{1}{2}\big(\cos(s-t) + \cos(s+t)\big)$

i) $\sin(s)\cos(t) = \frac{1}{2}\big(\sin(s+t) + \sin(s-t)\big)$

These formulas can, in turn, be derived algebraically from the more common formulas for the sine and cosine of a sum. I won't ask you to go through the proofs of every formula, but...

7. Prove formulas **c)**, **d)**, and **e)**.