Math 371, Spring 2006 — Homework #5

*Due **in class** on Monday, April 3*

1. Suppose that $\mathbf{x} = (1, 3, 5, 7, 9)$ and $\mathbf{y} = (2, 4, 6, 8)$. Compute the 5×4 matrix $\mathbf{x} \otimes \mathbf{y}$.

2. Suppose that $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\mathbf{z} \in \mathbb{R}^m$, and $\alpha, \beta \in \mathbb{R}$. Show by direct calculation that $(\alpha\mathbf{x} + \beta\mathbf{y}) \otimes \mathbf{z} = \alpha(\mathbf{x} \otimes \mathbf{z}) + \beta(\mathbf{y} \otimes \mathbf{z})$.

3. Suppose that $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$. Show by direct calculation that $\langle \mathbf{v} \otimes \mathbf{x}, \mathbf{w} \otimes \mathbf{y} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle \langle \mathbf{x}, \mathbf{y} \rangle$, where $\langle \cdot, \cdot \rangle$ represents the inner product of two vectors.

4. Let U, V, W be vector spaces. A function $f \colon U \times V \to W$ is said to be *bilinear* if for any $\mathbf{u}, \mathbf{u}_1, \mathbf{u}_2 \in U$, $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2 \in V$, and $\alpha, \beta \in \mathbb{R}$, $f(\alpha\mathbf{u}_1 + \beta\mathbf{u}_2, \mathbf{v}) = \alpha f(\mathbf{u}_1, \mathbf{v}) + \beta f(\mathbf{u}_2, \mathbf{v})$ and $f(\mathbf{u}, \alpha\mathbf{v}_1 + \beta\mathbf{v}_2) = \alpha f(\mathbf{u}, \mathbf{v}_1) + \beta f(\mathbf{u}, \mathbf{v}_2)$. (That is, f is linear in each argument separately.) In this problem, I will write $\mathbb{R}^n \otimes \mathbb{R}^m$ to mean the vector space $\mathbb{R}^{nm}$ considered as the tensor product of the vector spaces $\mathbb{R}^n$ and $\mathbb{R}^m$.

- **a)** Suppose $f \colon U \times V \to W$ is a bilinear function whose image is non-trivial (that is, there are vectors $\mathbf{u}, \mathbf{v}$ such that $f(\mathbf{u}, \mathbf{v}) \neq \mathbf{0}$.) Show that f is **not** a linear function on $U \times V$.
- **b)** Suppose $f \colon \mathbb{R}^n \times \mathbb{R}^m \to V$ is a bilinear function. Show that there is a unique linear function $\overline{f} \colon \mathbb{R}^n \otimes \mathbb{R}^m \to V$ such that $\overline{f}(\mathbf{x} \otimes \mathbf{y}) = f(\mathbf{x}, \mathbf{y})$ for all $\mathbf{x} \in \mathbb{R}^n, \mathbf{y} \in \mathbb{R}^m$. [Note: The problem here is that not every element of $\mathbb{R}^n \otimes \mathbb{R}^m$ is of the form $\mathbf{x} \otimes \mathbf{y}$, so the formula given here does not completely define $\overline{f}$. Hint: Work with the standard basis vectors.]
- **c)** Suppose that $F \colon \mathbb{R}^n \otimes \mathbb{R}^m \to V$ is a linear function. Show that there is a bilinear function $f \colon \mathbb{R}^n \times \mathbb{R}^n \to V$ such that $F = \overline{f}$.
- **d)** Deduce that there is a one-to-one correspondence between linear functions from $\mathbb{R}^n \otimes \mathbb{R}^m$ to V and bilinear functions from $\mathbb{R}^n \times \mathbb{R}^m$ to V. (This is, in some sense, the correct meaning of the tensor product of two vector spaces.)

5. Tensor products can be defined for complex vector spaces in the same way as for real vector spaces. When we studied the Discrete Fourier Transform, we used a basis of $\mathbb{C}^n$ defined in terms of n^{th} roots of unity. In particular, writing $\omega = e^{2\pi i/n}$, we defined a basis $\{\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_{n-1}\}$ where for $k = 0, 1, \ldots, n-1$,

$$\begin{aligned} \mathbf{w}_k &= (\omega^{0k}, \omega^{1k}, \omega^{2k}, \ldots, \omega^{(n-1)k}) \\ &= (1, e^{2\pi i k/n}, e^{2\pi i 2k/n}, \ldots, e^{2\pi i (n-1)k/n}) \end{aligned}$$

For a signal $\mathbf{s} = (s_1, s_2, \ldots, s_n) \in \mathbb{C}^n$, the inner product $\langle \mathbf{s}, \mathbf{w}_k \rangle$ gives the frequency-k component of $\mathbf{s}$, For a two-dimensional DFT, we would construct a basis of $\mathbb{C}^{nm}$ by taking bases of this form for $\mathbb{C}^n$ and $\mathbb{C}^m$ and forming tensor products. Describe the resulting basis of $\mathbb{C}^{nm}$, writing one typical basis element as an $n \times m$ matrix.