Math 375, Fall 2005 — About the Third Test

The third test for this course will be given in class on Friday,, December 2. It will cover Chapters 9, 10, 11, and 12, and the beginning of Chapter 13 (through page 120). These chapters cover closely related material on cosets, normal subgroups, homomorphisms, and isomorphisms.

There will not be any questions on the test about the latter part of Chapter 10, which concerned conjugacy classes and the class equation. There will be no questions on equivalence relations in general; however, there might be questions that require knowledge of the specific equivalence relations that define cosets.

Here are some important things you should know about:

Equivalence relation

Equivalence class

Right coset, $Hg = \{\, hg \mid h \in G \,\}$, of a subgroup H in a group G

$Ha = Hb$ iff $a \in Hb$ iff $ab^{-1} \in H$

For any $a, b \in G$, the cosets Ha and Hb have the same number of elements

Right cosets that are not identical are disjoint. (If $Ha \neq Hb$, then $Ha \cap Hb = \emptyset$.)

Coset notation for additive groups: $H + x$

Left cosets

Computing cosets for specific groups

Lagrange's Theorem: If G is a finite group and H is a subset of G, then $|H|$ divides $|G|$

Proof of Lagrange's Theorem using facts about cosets

If G is a finite group and $x \in G$, then $o(x)$ divides $|G|$

The index $[G : H]$ of a subgroup in a group

If p is a prime, then any group of order p is cyclic and is isomorphic to $\mathbb{Z}_p$

Normal subgroups

A subgroup H of G is normal iff for all $g \in G$ and $h \in H$, $ghg^{-1} \in H$

A subgroup H of G is normal iff for all $g \in G$, $Hg = gH$

Notation: $H \lhd G$ means that H is a normal subgroup of G

If G is any group, the center $Z(G)$ is a normal subgroup of G

If G is an abelian group, then any subgroup of G is normal in G

If G is any group, $g \in G$, and H is a subgroup of G, then gHg^{-1} is a subgroup of G

If G is any group and $H \lhd G$ then G/H is the set of right cosets of H in G

G/H is a group under the operation $*$ given by $Ha * Hb = H(ab)$

G/H is called a *quotient group* or *factor group*

If G is finite, then $|G/H| = [G : H] = |G|/|H|$

Any subgroup of index 2 is normal. Example: A_n is a normal subgroup of S_n

Homomorphisms, isomorphisms, and automorphisms

A homomorphism is a function $\varphi\colon G \to H$ such that $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in G$

Properties of homomorphism: $\varphi(e_G) = e_H$, $\varphi(x^k) = \varphi(x)^k$, if $o(x) < \infty$ then $o(\varphi(x)) \,|\, o(x)$, ...

The composition of homomorphisms is a homomorphism

Notation: $G \cong H$ means that G is isomorphic to H, that is, there exists an isomorphism from G to H

If φ is an isomorphism, then so is the inverse function φ^{-1}

$\mathrm{Aut}(G)$, the group of automorphisms of G

Kernel of a homomorphism $\varphi\colon G \to H$, $\mathrm{Ker}(\varphi) = \varphi^{-1}(e_H) = \{g \in G \,|\, \varphi(g) = e_H\}$

$\mathrm{Ker}(\varphi)$ is a normal subgroup of G

First Isomorphism Theorem: If $\varphi\colon G \to H$ is an onto homomorphism and $K = \mathrm{Ker}(\varphi)$, then $G/K \cong H$

A homomorphism $\varphi\colon G \to H$ is one-to-one iff $\mathrm{Ker}(\varphi) = \{e_G\}$

Homomorphisms from $\mathbb{Z}$ to a group G

Homomorphisms from $\mathbb{Z}_n$ to a group G

$\mathrm{Aut}(\mathbb{Z}_n) \cong U(n)$

The subgroup $n\mathbb{Z}$ of $\mathbb{Z}$; cosets of $n\mathbb{Z}$ in $\mathbb{Z}$

$\mathbb{Z}_n \cong \mathbb{Z}/n\mathbb{Z}$

The determinant is a homomorphism from $GL(2, \mathbb{R})$ to $(\mathbb{R}^*, \cdot)$ whose kernel is $SL(2, \mathbb{R})$

$(\mathbb{R}, +) \cong (\mathbb{R}^+, \cdot)$

$D_n = \{e, f, f^2, \ldots, f^{n-1}, g, gf, gf^2, \ldots, gf^{n-1}\}$, where $f^n = e$, $g^2 = e$, and $gf^k = f^{-k}g$ for any integer k