This homework is due on Friday, September 5. It is mainly about properties of the modulus and the conjugate and some applications. Remember that you can work on homework problems with other people in the class, but you should write up your own careful solutions. Remember to show all your work! The last problem is more difficult than the first five.

1. Check by direct calculation that for two complex numbers z and w, $\overline{z+w} = \overline{z}+\overline{w}$ and $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$. Deduce that for any polynomial $P(z)$ with real coefficients, $\overline{P(z)} = P(\overline{z})$.

2. Suppose that $P(z)$ is a polynomial with real coefficients. Use the previous exercise to show that a complex number w is a zero of $P(z)$ if and only if $\overline{w}$ is a zero of $P(z)$. Suppose that $w = a + ib$, where a and b are real. Check that $P(z) = (z - w)(z - \overline{w})$ has real coefficients, and compute them in terms of a and b.

3. The Fundamental Theorem of Algebra says that any polynomial $P(z)$ with complex coefficients can be factored into a constant term and factors of the form $(z - w)$ for complex zeros w of the polynomial. Suppose that $P(x)$ is a polynomial with real coefficients. Based on the Fundamental Theorem and the previous exercise, what can be said about factoring $P(x)$ into factors that have real coefficients? (Justify your answer!)

4. Check by direct calculuations using rectangular coordinates that $|zw| = |z| \cdot |w|$ for complex numbers z and w. Deduce that $|z^n| = |z|^n$ for a positive integer n.

5. (Exercise 9a in the textbook.) It is an interesting fact that the product of two sums of squares is itself a sum of squares. That is, if that a, b, c, and d are integers, then there exist integers u and v such that $(a^2 + b^2)(c^2 + d^2) = (u^2 + v^2)$. Prove this using complex numbers. (Hint: See the previous problem.)

6. (Exercise 20a in the textbook.) Let $P(z) = 1 + 2z + 3z^2 + \cdots + nz^{n-1}$. By considering $(1 - z)P(z)$, show that all the zeros of $P(z)$ are in the unit disk, $|z| \leq 1$.