This homework is due Friday, September 12

1. Use the Cauchy-Riemann equations to decide whether each of the following polynomials is analytic. If it is analytic, find a complex polynomial $Q(z)$ such $P(x,y) = Q(x+iy)$.

a) $P_1(x,y) = (x^3 - 3xy^2) + i(y^3 - 3x^2y)$

b) $P_2(x,y) = (x^4 + y^4 - 6x^2y^2) + i(4x^3y - 4xy^3)$

c) $P_3(x,y) = (x - 2x^2 + 2y^2) + i(y - 4xy)$

2. Suppose that $f(z)$ is a complex-valued function defined on a subset S of $\mathbb{C}$. Let $z_o \in S$. Write a careful proof that the following are equivalent:

a) For any sequence $z_1, z_2, z_3, \dots$ of points in S, if z_n converges to z_o, then $f(z_n)$ converges to $f(z_o)$.

b) For any $\varepsilon > 0$, there is a $\delta > 0$ such that for any $z \in S$, if $|z - z_o| < \delta$ then $|f(z) - f(z_o)| < \varepsilon$.

3. Suppose that X and Y are connected subsets of $\mathbb{C}$ and that $X \cap Y$ is not empty. Show that $X \cup Y$ is connected (using the definition of connected set).

4. Suppose that $\{a^n\}$ is a sequence of non-negative real numbers and that $\overline{\lim\limits_{n\to\infty}}\, a_n = L$, where $L < \infty$.

a) Show that $\{a_n\}$ has a subsequence that converges to L.

b) Suppose that $M > L$. Can $\{a_n\}$ have a subsequence that converges to M? Find an example or show that no example exists.

c) Suppose that $M < L$. Can $\{a_n\}$ have a subsequence that converges to M? Find an example or show that no example exists.

(Note: A subsequence of $a_1, a_2, a_3, \dots$ is a sequence $a_{n_1}, a_{n_2}, a_{n_3}, \dots$ where $n_1 < n_2 < n_3 \cdots$. A subsequence can often be defined inductively; that is, given the first N terms of the subsequence, show how to find the $(N+1)^{\text{st}}$ term.)

5. Suppose that $f(z)$ is a complex-valued function defined on a neighborhood of $a \in \mathbb{C}$. Suppose that f is differentiable at a, and that $f'(a) \neq 0$. Let $g(z) = f(\overline{z})$. Note that $g(\overline{a}) = f(\overline{\overline{a}}) = f(a)$ and that g is defined on a neighborhood of $\overline{a}$. Show that g is **not** differentiable at $\overline{a}$, using the definition of the derivative.