This homework is due Wednesday, September 17

1. The nice thing about the root test with lim sup is that it apples to any series. But it's not always easy to calculate. Recall the *ratio test*: Consider a series $\sum_{k=0}^{\infty} a_k$. If $\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|}$ exists and is less than 1, then the series converges absolutely. If the limit exists and is greater than 1, then the series diverges. The proof is the same for complex series as for real series. (You do not have to prove it!)

a) Now, consider the power series $\sum_{k=0}^{\infty} c_k z^k$. Suppose that $\lim_{k\to\infty} \frac{|c_{k+1}|}{|c_k|}$ exists and is equal to 0. Show that the power series converges for all z.

b) Now, suppose that the limit exists and is equal to L, where $0 < L < \infty$. Let $R = \frac{1}{L}$. Show that the power series converges for $|z| < R$ and diverges for $|z| > R$.

c) Use the results from **a)** and **b)** to find the radii of convergence of $\sum_{k=0}^{\infty} \frac{\sqrt{k} \cdot z^k}{k!}$ and $\sum_{k=0}^{\infty} \frac{3^k z^k}{\sqrt{2k+1}}$.

2. The series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges for all z. Let $f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$. Let $x \in \mathbb{R}$ and consider $f(ix)$. Break the series for $f(ix)$ down into its real and imaginary parts. That is, write $f(ix) = \sum_{n=0}^{\infty} a_n x^n + i \sum_{n=0}^{\infty} b_n x^n$ where the a_n and b_n are real. Do you recognize these series? (If not, look up the series for $\sin(x)$ and $\cos(x)$.)

3. We know that $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ for $|z| < 1$. Taking the derivative gives $\sum_{n=1}^{\infty} n z^{n-1} = \frac{1}{(1-z)^2}$. This question asks you to do some manipulation of power series to find formulas for several other series. (The real goal is part **d)**. Pay attention to the starting value for n.)

a) Find a formula for $\sum_{n=1}^{\infty} n z^n$

b) Find a formula for $\sum_{n=2}^{\infty} n(n-1) z^{n-1}$

c) Find a formula for $\sum_{n=2}^{\infty} n^2 z^{n-1}$

d) Find a formula for $\sum_{n=1}^{\infty} n^2 z^n$