*This homework is due Wednesday, October 1*

**1.** Let $C$ be the line segment from the point 1 to the point $i$. By observing that the midpoint of the line segment is the closest point on the line segment to 0, and hence has the smallest modulus, show that

$$\left|\int_C \frac{1}{z^4}\,dz\right| \;<\; 4\sqrt{2}$$

without evaluating the integral.

**2.** [Exercise 4.11 and 4.12 from the book—but no answers in the back.]

**a)** Suppose that $f(z)$ is an analytic function on a convex region D, and that $|f'(z)| < 1$ for $z \in D$. Prove that $f$ is a "contraction" on D. That is, $|f(z) - f(w)| < |z - w|$ for all $z, w \in D$. (Note: A region is convex if for any two points in the region, the line segment between the two points lies entirely in the region.)

**b)** Show that for any $z$ and $w$ in the left half-plane ($x < 0$), $|e^z - e^w| < |z - w|$.

**3.** [Path independence.] Suppose that $f(z)$ is an entire function. Suppose $C_1$ and $C_2$ are smooth curves whose endpoints are the same. (That is, if $C_1$ is given by $z_1 : [a, b] \to \mathbb{C}$ and $C_2$ is given by $z_2 : [c, d] \to \mathbb{C}$, then $z_1(a) = z_2(c)$, and $z_1(b) = z_2(d)$.) Show that

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

(This is easy. Use Propositions 4.12 and 4.15.)

**4.** Define $f(z) = f(x + iy) = log|x + iy| + i \arctan(y/x)$, for $z$ in the right half-plane, $x > 0$. (Note that $\arctan(y/x)$ is one of the possible values of $\operatorname{Arg}(x + iy)$, so $f(z)$ is one of the many possible "complex logarithms" of $z$, for $z$ in the right half-plane. In particular, $e^{f(z)} = z$.)

**a)** Use the Cauchy-Riemann equations to check directly that $f$ is analytic.

**b)** Check that $f'(z) = 1/z$ for all $z$ in the right half-plane.

**c)** Check that $f(e^z) = z$, for all $z$ such that $-\frac{\pi}{2} < \operatorname{Im}(z) < \frac{\pi}{2}$.

**d)** Find $\displaystyle\int_C \frac{1}{z}\,dz$, where $C$ is any path in the right half-plane from the point 1 to the point $1 + i$.

**5.** We know that a convergent power series with radius of convergence $R > 0$ defines an analytic function on the domain $|z| < R$. We also know that the series converges *uniformly* on disks $|z| < r$ for $0 < r < R$. (See the remark after Theorem 2.8 on page 27.)

**a)** Using Proposition 4.11, show that a power series can be *integrated* term by term to get a series that converges to its antiderivative on the domain $|z| < R$.

**b)** The series $\displaystyle\sum_{k=0}^{\infty}(-1)^k z^k$ converges to $f(z) = \dfrac{1}{1+z}$ for $|z| < 1$. Integrate the series term-by-term to obtain a series for a function $F(z)$ such that $F'(z) = f(z)$ for $|z| < 1$.