This is a take-home test. You should not work with anyone on test, and you should not discuss the problems with anyone except your professor. You can consult class notes and the textbook. You should not use other books, the Internet, or any other resources.

Be verbose enough to make me believe your answers!

1. Show that the function $f(z) = f(x+i\,y) = (xe^x \cos y - ye^x \sin y) + i\,(xe^x \sin y + ye^x \cos y)$ is entire.

2. Let M be the set of two-by-two matrices of real numbers. Define a function $\Phi\colon \mathbb{C} \to M$ by $\Phi(x+iy) = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$. Show that $\Phi(zw) = \Phi(z)\Phi(w)$, that $\Phi(z+w) = \Phi(z) + \Phi(w)$, and that $\Phi(z^{-1}) = \Phi(z)^{-1}$. (Thus, the complex numbers could be defined as a certain set of real matrices with the usual multiplication and addition.)

3. Suppose that the functions $f(z)$ and $\overline{f(z)}$ are both analytic on a domain D. Show that f is constant on D.

4. Suppose that a is a real number and that $f(z)$ is an analytic function on an open neighborhood $D(a;r)$ of a. Suppose that f maps real numbers to real numbers. (That is, for any real number b in $D(a;r)$, $f(b)$ is real.)

a) Show that in the power series $f(z) = \sum_{k=0}^{\infty} c_k (z-a)^k$ for f about a, all the coefficients c_k are real numbers. (Hint: What can you say about $f^{(k)}(z)$?)

b) Conclude that $f(\overline{z}) = \overline{f(z)}$ for $z \in D(a;r)$.

5. The complex logarithm is a "multi-valued function," given by $\operatorname{Log}(z) = \ln|z| + i\operatorname{Arg}(z)$. If w is one value of $\operatorname{Log}(z)$, then the full set of values is $\{w + 2\pi i k \mid k \in \mathbb{Z}\}$.

a) For an integer $n > 1$, the function $z^{1/n}$ is also multivalued. Show that $z^{1/n}$ can be expressed as $e^{\operatorname{Log}(z)/n}$, and explain why this formula is n-valued.

b) We can define z^w for any complex numbers z and w with $z \neq 0$ to be $e^{w\operatorname{Log}(z)}$, with the understanding that this formula can represent many values. Find all the values of 1^i.

c) Find all the values of $1^{1/\pi}$, and describe what the set looks like as a subset of the complex plane.

d) Suppose that we apply the formula for z^w to z^2. Explain why the resulting formula for z^2 is correct and why it is single-valued.

6. Suppose that $f(z)$ is an entire function and that $|f(z)| \leq |e^z|$ for all z. What can you conclude about f? Prove your answer. (Hint: e^z is never zero.)