LAB Week 10

MATH 130 Section 2
April 4, 2019
Covering Sections 3.9, 3.10 and 3.11

Your Name (Print): ANSWER KEY

1. Find $(f^{-1})'(3)$ if $f(x) = \sqrt{x-4}$ **without** finding the inverse function.

$$(f^{-1})'(3) = \frac{1}{f'(f^{-1}(3))}$$

$$f^{-1}(3) = k \Rightarrow f(k) = 3$$

since $f(13) = \sqrt{13-4} = \sqrt{9} = 3$,
$k = 13$

$$f'(x) = \frac{1}{2}(x-4)^{-1/2} \cdot 1 = \frac{1}{2\sqrt{x-4}}$$

$$= \frac{1}{f'(13)}$$

$$= \frac{1}{\frac{1}{2\sqrt{13-4}}}$$

$$= \frac{1}{\frac{1}{2\sqrt{9}}}$$

$$= \frac{1}{\frac{1}{6}} = 6$$

2. Find the derivative of the inverse of $f(x) = \tan x$ using the theorem on inverse functions (Theorem 3.21 in your text, though you may want to use our notation, which is similar to that shown in the side margin). That is, use the theorem to derive a proof of the formula we know.

$f(x) = \tan x \Rightarrow f^{-1}(x) = \arctan x$ and $f'(x) = \sec^2 x$

$$\text{So } (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'(\arctan x)} = \frac{1}{\sec^2(\arctan x)}$$

Method 1

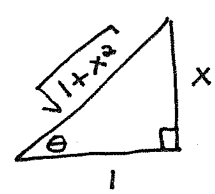
$\sec^2 x = 1 + \tan^2 x$

so $\sec^2(\arctan x) = 1 + \tan^2(\arctan x)$

$= 1 + x^2$ by properties of inverse functions

Method 2

$\arctan x = \theta \Rightarrow \tan\theta = \frac{x}{1} = \frac{\text{opp}}{\text{adj}}$



so $\sec\theta = \frac{\text{hyp}}{\text{adj}}$

$$= \frac{\sqrt{1+x^2}}{1}$$

$$= \sqrt{1+x^2}$$

$$\text{Therefore } (f^{-1})'(x) = \frac{1}{\sec^2(\arctan x)} = \frac{1}{1+x^2} = \frac{d}{dx}\arctan x$$

:)

3. Differentiate the following functions.

(a) $y = \sec(\arcsin(4^{2x^2}))$

$$\frac{dy}{dx} = \sec\left(\arcsin\left(4^{2x^2}\right)\right)\tan\left(\arcsin\left(4^{2x^2}\right)\right) \cdot \frac{1}{\sqrt{1-\left(4^{2x^2}\right)^2}} \cdot 4^{2x^2}\ln 4 \cdot 4x$$

(b) $y = x^{\log_5 x}$

$$\ln|y| = \ln\left|x^{\log_5 x}\right| = \log_5 x \cdot \ln|x|$$

$$\frac{1}{y}\frac{dy}{dx} = \log_5 x \cdot \frac{1}{x} + \frac{1}{x\ln 5} \cdot \ln|x|$$

$$\frac{dy}{dx} = y\left[\frac{\log_5 x}{x} + \frac{\ln|x|}{x\ln 5}\right]$$

$$\frac{dy}{dx} = x^{\log_5 x}\left[\frac{\log_5 x}{x} + \frac{\ln|x|}{x\ln 5}\right]$$

(c) $y = \dfrac{(4x^3 - 8x^2 + 1)^7 \cos^2 x}{\sqrt[5]{x^9 - 6x}}$ (Hint: Think about using the method you used for (b).)

$$\ln|y| = \ln\left|\frac{(4x^3-8x^2+1)^7\cos^2 x}{\sqrt[5]{x^9-6x}}\right|$$

$$= \ln\left|(4x^3-8x^2+1)^7\cos^2 x\right| - \ln\left|\sqrt[5]{x^9-6x}\right|$$

$$= \ln\left|(4x^3-8x^2+1)^7\right| + \ln|\cos^2 x| - \frac{1}{5}\ln|x^9-6x|$$

$$= 7\ln|4x^3-8x^2+1| + 2\ln|\cos x| - \frac{1}{5}\ln|x^9-6x|$$

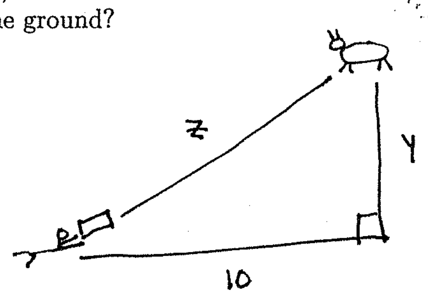
$$\frac{1}{y}\frac{dy}{dx} = 7 \cdot \frac{1}{4x^3-8x^2+1} \cdot (12x^2-16x) + 2 \cdot \frac{1}{\cos x} \cdot (-\sin x) - \frac{1}{5}\frac{1}{x^9-6x} \cdot (9x^8-6)$$

$$\frac{dy}{dx} = y\left[\frac{7(12x^2-16x)}{4x^3-8x^2+1} - 2\tan x - \frac{9x^8-6}{5(x^9-6x)}\right]$$

$$\frac{dy}{dx} = \frac{(4x^3-8x^2+1)^7\cos^2 x}{\ldots}\left[\frac{7(12x^2-16x)}{\ldots} - 2\tan x - \frac{9x^8-6}{5(x^9-6x)}\right]$$

4. As a prank, a bunch of students walked a cow upstairs to the third floor of Lansing Hall, which, as you well know, does not have an elevator. Cows do not walk **down** stairs so the administration had to rent a crane to lower the cow from a window. As the crane lowered the cow straight down at a rate of 2m/min, a student lying on the ground 10m from the building video-taped the cow's descent.

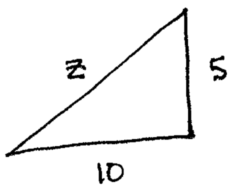
(a) How fast was the distance between the video-taper and the cow changing when the cow was 5m from the ground?



Know: $\frac{dy}{dt} = -2$ Want: $\left.\frac{dz}{dt}\right|_{y=5} = ?$

$$10^2 + y^2 = z^2$$

$$0 + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$



$$\cancel{2}(5)(-2) = \cancel{2}(\sqrt{125})\frac{dz}{dt}$$

$$\frac{-10}{\sqrt{125}} = \frac{dz}{dt}$$

$$\frac{-10}{5\sqrt{5}} = \frac{-2}{\sqrt{5}} = \frac{dz}{dt}$$

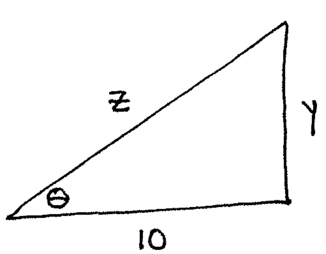
$$5^2 + 10^2 = z^2$$

$$25 + 100 = z^2$$

$$z = \sqrt{125}$$

(b) At what rate was the angle of elevation of the video camera changing when the distance between the student and the cow was 15m?

Know: $\frac{dy}{dt} = -2$ Want: $\left.\frac{d\theta}{dt}\right|_{z=15}$



$$\tan\theta = \frac{y}{10} = \frac{1}{10}y$$

$$\sec^2\theta\frac{d\theta}{dt} = \frac{1}{10}\frac{dy}{dt}$$

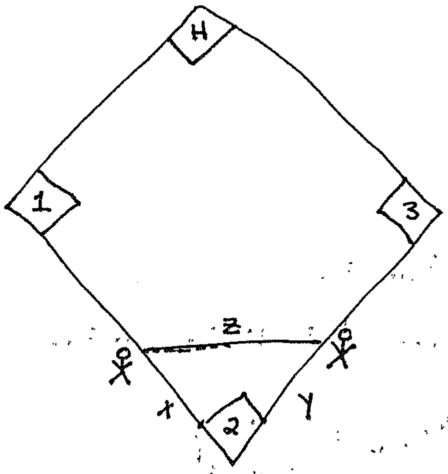
$$\sec\theta = \frac{H}{A} = \frac{15}{10}$$

$$\left(\frac{15}{10}\right)^2\frac{d\theta}{dt} = \frac{1}{10}(-2)$$

$$\frac{d\theta}{dt} = \frac{-2}{\cancel{10}}\cdot\frac{10^{\cancel{2}}}{15^2} = \frac{-2\cdot2\cdot\cancel{5}}{3^2\cdot5^{\cancel{2}}} = \frac{-4}{45}$$

$$= \frac{-4}{45}\ \text{rad/min}$$

5. Historical Reference: Think back to the fall of 2013. It is game 6 of the World Series and neither the Cardinals nor the Red Sox have scored by the bottom of the third. But after Jonny Gomes is hit by a pitch, Jacoby Ellsbury is on 3rd, David Ortiz is on 2nd and Gomes is on first – bases loaded as Shane Victorino comes to the plate. At the moment Victorino hits the ball, Gomes runs to second base at 20 ft/sec; simultaneously Ortiz runs to third base at 18 ft/sec. How fast is the distance between Gomes and Ortiz changing 1 second after the ball is hit? (Note: that the bases lie on the corners of a square and the distance between consecutive bases is 90 ft. Also note I have no idea if this was their actual speed. Lastly, after the play, the Red Sox were ahead by three runs...and they never lost the lead.)



Know: $\frac{dx}{dt} = -20$ Want: $\left.\frac{dz}{dt}\right|_{t=1}$

$\frac{dy}{dt} = 18$

$$x^2 + y^2 = z^2$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

After 1 second Gomes has run 20 feet making $x = 90-20=70$ and Ortiz has run 18 feet making $y = 18$.

$$70^2 + 18^2 = z^2$$
$$4900 + 324 = z^2$$
$$5224 = z^2$$
$$z = \sqrt{5224} \approx 72.3$$

$$\cancel{2}(-20)(70) + \cancel{2}(18)(18) = \cancel{2}(\sqrt{5224})\frac{dz}{dt}$$

$$\frac{-1400 + 324}{\sqrt{5224}} = \frac{dz}{dt}$$

$$\frac{-1076}{\sqrt{5224}} = \frac{dz}{dt}$$

$$\frac{dz}{dt} \approx -14.89 \text{ ft/sec}$$

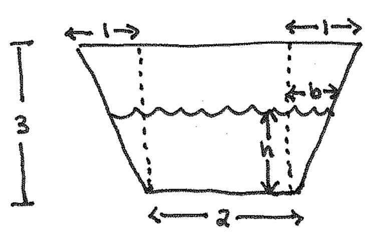
6. Reread the proof of Theorem 3.21 on pages 223-224. Copy the proof here and write in any missing details. Make sure you understand the process. Isn't that neat?!

See proof on pages 223-224. :)

7. A water trough is 10 meters long and a cross-section has the shape of an isoceles trapezoid that is 2 meters wide at the bottom, 4 meters wide at the top, and has a height of 3 meters. If the trough is being filled with water at the rate of 10 meters^3/minute, how fast is the water level rising when the water is 2 meters deep?

Set up:

End of trough:



Know: $\frac{dV}{dt} = 10 \ m^3/min$

Want: $\left.\frac{dh}{dt}\right|_{h=2}$

Where V = the volume of the water.

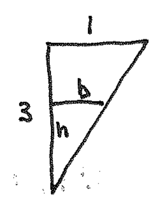
$V = (\triangledown + \square + \triangledown) \cdot 10$

$= (\frac{1}{2}bh + 2h + \frac{1}{2}bh) \cdot 10$

$= (bh + 2h)10$

$= ((\frac{1}{3}h)h + 2h)10$

$= \frac{10}{3}h^2 + 20h$



$\frac{b}{h} = \frac{1}{3}$

so $b = \frac{1}{3}h$

$\frac{dV}{dt} = \frac{20}{3}h\frac{dh}{dt} + 20\frac{dh}{dt}$

$10 = \left[\frac{20}{3}(2) + 20\right]\frac{dh}{dt}$

$10 = \left[\frac{40}{3} + 20\right]\frac{dh}{dt}$

$10 = \left[\frac{100}{3}\right]\frac{dh}{dt}$

$\frac{dh}{dt} = 10 \cdot \frac{3}{100} = \frac{3}{100} \ m/min$

ANSWERS not SOLUTIONS:

8. Differentiate the following and simplify your answer: $y = (\ln x)^x$.

$$\frac{dy}{dx} = (\ln x)^x \left[\frac{1}{\ln x} + \ln|\ln x|\right]$$

9. A kite 100 feet above the ground moves horizontally at a speed of 8 ft/s. At what rate is the angle between the string and the horizontal decreasing when 200ft of string have been let out?

$$-\frac{1}{50} \text{ rad/s}$$

10. Find the derivatives of the following functions. Simplify your answers.

(a) $f(\theta) = \arctan(\cos\theta)$

$$f'(\theta) = \frac{-\sin\theta}{1+\cos^2\theta}$$

(b) $y = e^{\text{arcsec}t}$

$$y' = \frac{e^{\text{arcsec}t}}{|t|\sqrt{t^2-1}}$$

(c) $g(x) = (\arcsin x)^2$

$$g'(x) = \frac{2\arcsin x}{\sqrt{1-x^2}}$$

(d) $h(x) = \arcsin(x^2)$

$$h'(x) = \frac{2x}{\sqrt{1-x^2}}$$

11. If $f(x) = 3 + x + e^x$, find $(f^{-1})'(4)$.

$$(f^{-1})'(4) = \frac{1}{2}$$