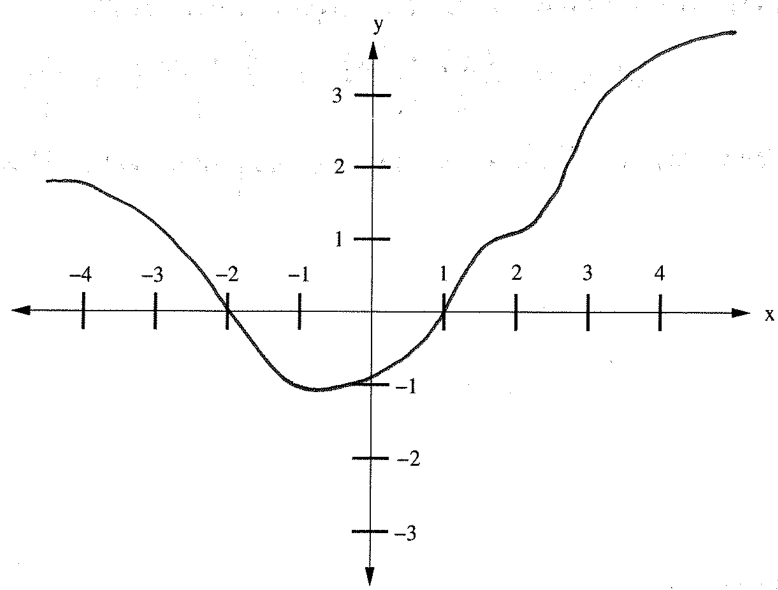
LAB Week 12

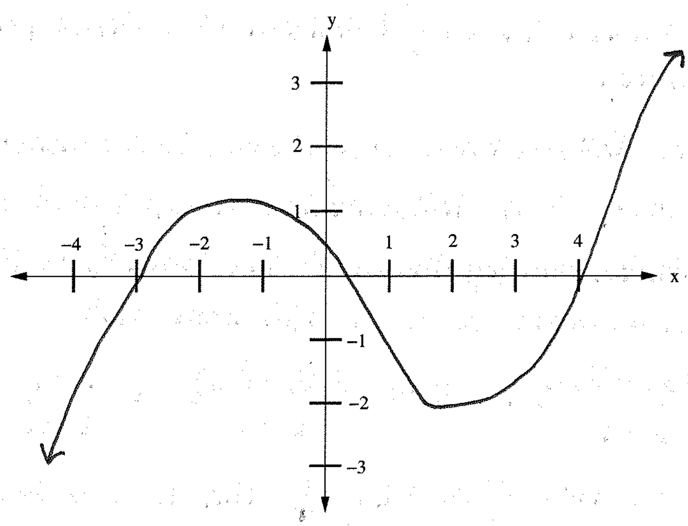
MATH 130 Section 2
April 18, 2019
Covering Sections 4.1-4.4

Your Name (Print): ANSWER KEY

1. Sketch the graph of a function f that is continuous on $[-3, 4]$ and has the following properties: critical numbers at $x = -1$ and $x = 2$, a local minimum at $x = -1$, but no local extrema at $x = 2$.



2. Sketch the graph of a function g that has a local minimum and a local maximum, but does not have an absolute minimum nor an absolute maximum.



2. Show there does not exist a differentiable function on $[1,5]$ with $f(1) = -3$, $f(5) = 9$, and $f'(x) \leq 2$ for all x. (Hint: Use the Mean Value Theorem!)

Since f is differentiable on $[1,5]$, it is continuous on $[1,5]$ and so the Mean Value Theorem may be applied to f on $[1,5]$.

Thus there exists a c in $(1,5)$ such that

$$f'(c) = \frac{f(5)-f(1)}{5-1} = \frac{9-(-3)}{4} = \frac{12}{4} = 3$$

But then $f'(c) > 2$ so we may not have $f'(x) \leq 2$ for all x.

3. A number a is called a **fixed point** of a function f if $f(a) = a$. Prove that if f is differentiable everywhere and $f'(x) \neq 1$ for all real numbers x, then f has at most one fixed point. (This is challenging, but here are some hints: use proof by contradiction like we did for the exactly one real root problem, and then think about how the Mean Value Theorem might help you. This is neat!)

Proof by contradiction:

Let f be a function that is differentiable everywhere and $f'(x) \neq 1$ for all real numbers x. Suppose f has two fixed points a and b such that $a < b$. Then by definition of a fixed point, $f(a) = a$ and $f(b) = b$.

Since f is differentiable everywhere, it is continuous everywhere. In particular, f is differentiable on (a,b) and continuous on $[a,b]$. Thus f satisfies the hypotheses of the Mean Value Theorem on $[a,b]$, and so there exists a c in (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$ But $$\frac{f(b)-f(a)}{b-a} = \frac{b-a}{b-a} = 1,$$

and so we have $f'(c) = 1$. $\lightning$ This is a contradiction since $f'(x) \neq 1$ for all real numbers x.

Hence f has at most one fixed point.

$\blacksquare$

6. Let $f(x) = \dfrac{x^2 - 4}{x^2 + 3}$.

(a) Find the intervals on which f is increasing and decreasing.

$$f'(x) = \frac{(x^2+3)(2x) - (x^2-4)(2x)}{(x^2+3)^2} = \frac{2x\left[x^2+3 - x^2 + 4\right]}{(x^2+3)^2}$$

$$= \frac{14x}{(x^2+3)^2}$$

$f'(x) = 0$ when $14x = 0 \Rightarrow x = 0$

$f'(x)$ always exists since $x^2+3 \geq 3$ for all x

	$(-\infty, 0)$	0	$(0, \infty)$
$f'(x)$	$- - - - - - - -$	0	$+ + + + + + +$
$14x$	$- - - - - - - -$	0	$+ + + + +$
$(x^2+3)^2$	$+ + + + +$		$+ + + +$

f is increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$.

(b) Find all local maximum and minimum values of f (be sure to state whether they are minimum or maximum), if they exist.

As can be seen from the First Derivative Test above,
f has a local minimum at $(0, -\frac{4}{3})$.

$f(0) = \dfrac{-4}{3}$

(c) Find all intervals on which f is concave up or down.

$$f'(x) = \frac{14x}{(x^2+3)^2}$$

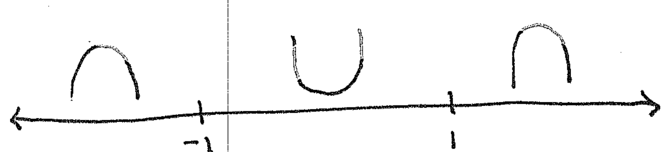
$$f''(x) = \frac{(x^2+3)^2(14) - (14x)(2(x^2+3)\cdot 2x)}{(x^2+3)^4}$$

$$= \frac{\cancel{(x^2+3)}\left[14x^2+42-56x^2\right]}{(x^2+3)^{\cancel{4}\,3}} = \frac{42-42x^2}{(x^2+3)^3}$$

$$= \frac{42(1-x^2)}{(x^2+3)^3} = \frac{42(1-x)(1+x)}{(x^2+3)^3}$$

$f''(x) = 0$ when $42(1-x)(1+x) = 0 \Rightarrow x = \pm 1$

$f''(x)$ always exists since $x^2+3 \geq 3$ for all x



	$x<-1$	-1	$-1<x<1$	1	$x>1$
$f''(x)$	$- - - - -$	0	$+ + + + +$	0	$- - - -$
$42(1-x)$	$+ + +$		$+ + +$	0	$- - -$
$1+x$	$- - - -$	0	$+ + + +$		$+ + +$
$(x^2+3)^3$	$+ + +$		$+ + +$		$+ + +$

f is concave up on $(-1,1)$ and concave down on $(-\infty,-1)$ and $(1,\infty)$

(d) Find all points of inflection, if they exist.

As can be seen above, f has inflection points at $(-1, -\frac{3}{4})$ and $(1, -\frac{3}{4})$.

$$f(-1) = \frac{(-1)^2-4}{(-1)^2+3} = \frac{1-4}{1+3} = \frac{-3}{4} = f(1)$$

ANSWERS not SOLUTIONS

5. Section 4.4 is the culmination of a lot of the material we have been studying so far this semester. We are now able to put together pieces of information about a function to graph it. Your reading assignment for Wednesday asked that you compile all the information into a table. Here is a question that uses that information. There will be a question similar to this on the exam next week! You will certainly need another piece of paper to do this question!

Let $f(x) = \dfrac{x+1}{\sqrt{x^2+1}}$.

State both coordinates for all requested points.

(a) Find the domain of f. Use interval notation to state your solution.

$(-\infty, \infty)$

(b) Find all x and y intercepts. Label which is which.

y-intercept: $(0,1)$

x-intercept: $(-1,0)$

(c) Find all horizontal asymptotes. State the asymptotes explicitly. If no horizontal asymptotes exist, explain why.

$y = 1$ and $y = -1$

(d) Find all vertical asymptotes and related information. State the asymptotes explicitly. If no vertical asymptotes exist, explain why.

NONE

(e) Find all intervals on which f is increasing or decreasing.

f is increasing on $(-\infty, 1)$

f is decreasing on $(1, \infty)$

(f) Find all local extrema, if they exist. (Recall that you need to include x and y values so that you can plot them on your graph.) If no local extrema exist, explain why.

local maximum at $(1, \sqrt{2})$

(g) Find all intervals on which f is concave up or down. (Note: It is ok if your numbers are complicated! Leave in exact form!)

f is concave up on $\left(-\infty, \frac{3-\sqrt{17}}{4}\right)$ and $\left(\frac{3+\sqrt{17}}{4}, \infty\right)$

f is concave down on $\left(\frac{3-\sqrt{17}}{4}, \frac{3+\sqrt{17}}{4}\right)$

(h) Find all points of inflection, if they exist. (Recall that you need to include x and y values so that you can plot them on your graph.) If no inflection points exist, explain why.

$\approx (1.78, 1.36)$ and $(-0.28, 0.69)$

(i) Plot points, sketch asymptotes and sketch the graph of $f(x)$ using the above information. Label clearly (coordinates, asymptote names, etc.). Think carefully about the scale before you start drawing. That is, choose a scale that will make the graph's features clear.