LAB Week 4

MATH 130 Section 3
February 14, 2018: Happy Valentine's Day!
Covering Sections 2.7 and 2.3

Your Name (Print): ANSWER KEY

1. What is the value of $\lim_{x\to 12}\left(10-4x\right)$? Use the ϵ-δ definition of a limit to prove your assertion. (Refer to the outline on the worksheet passed out in class on Friday. Remember words are important!)

$$\lim_{x\to 12}(10-4x) = 10-4(12) = -38$$

Scratch work to guess δ:

Given $\epsilon > 0$ we want to find $\delta > 0$ such that if $0<|x-12|<\delta$, then $|(10-4x)-(-38)|<\epsilon$.

$$|(10-4x)-(-38)|<\epsilon \Rightarrow |10-4x+38|<\epsilon$$
$$\Rightarrow |-4x+48|<\epsilon$$
$$\Rightarrow |-4||x-12|<\epsilon$$
$$\Rightarrow |x-12|<\frac{\epsilon}{4}.$$

Guess $\delta=\frac{\epsilon}{4}$.

Proof:

Given $\epsilon>0$, let $\delta=\frac{\epsilon}{4}$.

Then if $0<|x-12|<\delta$, then

$$\begin{aligned}|(10-4x)-(-38)| &= |-4x+48| \\ &= |-4||x-12| \\ &= 4|x-12| \\ &< 4\delta = \cancel{4}\left(\frac{\epsilon}{\cancel{4}}\right)=\epsilon.\end{aligned}$$

Thus $|(10-4x)-(-38)|<\epsilon$ and therefore

$$\lim_{x\to 12}(10-4x) = -38.$$

$\blacksquare$

☞ **Good limit grammar.** Remember to use **equal signs** and **limit symbols** in your limit calculations. When numerically evaluating the limit in the last step(s) (i.e. when you have plugged in!), do not use the limit symbol because you have found the limit. For example,

$$\lim_{x\to -4} \sqrt[3]{3x-15} \overset{\text{Frac Pow}}{=} \sqrt[3]{\lim_{x\to -4} 3x-15} \overset{\text{Poly}}{=} \sqrt[3]{-27} = -3.$$

2. Suppose that $\lim_{x\to 1} f(x) = 4$, $\lim_{x\to 1} g(x) = -3$ and $\lim_{x\to 1} h(x) = 5$. Evaluate these limits, use good limit grammar, and indicate the properties that you used as in the example above. Show one step at a time!

(a) $\lim_{x\to 1}[2g(x) - f(x)x^3] \overset{\text{Diff}}{=} \lim_{x\to 1} 2g(x) - \lim_{x\to 1} f(x)x^3$

$\overset{\text{Cons. Mult.}}{\underset{\text{Prod.}}{=}} 2\lim_{x\to 1} g(x) - \left[\lim_{x\to 1} f(x) \cdot \lim_{x\to 1} x^3\right]$

$\overset{\text{Poly}}{=} 2(-3) - \left[4 \cdot 1^3\right]$

$= -6 - 4$

$= -10$

(b) $\lim_{x\to 1} \dfrac{g(x)+6x}{\sqrt[3]{f(x)h(x)}} \overset{\text{Quot.}}{=} \dfrac{\lim_{x\to 1}(g(x)+6x)}{\lim_{x\to 1}\sqrt[3]{f(x)h(x)}}$

$\overset{\text{Sum}}{\underset{\text{Frac Pow}}{=}} \dfrac{\lim_{x\to 1} g(x) + \lim_{x\to 1} 6x}{\sqrt[3]{\lim_{x\to 1} f(x)h(x)}} \overset{\text{Prod}}{=} \dfrac{\lim_{x\to 1} g(x) + \lim_{x\to 1} 6x}{\sqrt[3]{\lim_{x\to 1} f(x) \cdot \lim_{x\to 1} h(x)}}$

$\overset{\text{Poly}}{\underset{\text{(Thm 2.4)}}{=}} \dfrac{(-3)+6(1)}{\sqrt[3]{4\cdot 5}} = \dfrac{3}{\sqrt[3]{20}}$

3. Evaluate the following limits, if they exist. Remember good limit grammar! If the limit does not exist, explain why.

(a) $\lim_{x\to -1} \dfrac{x^3-x}{x^2-5x-6}$ (Note: looks like $\frac{0}{0} \Rightarrow$ DO MORE WORK!)

$= \lim_{x\to -1} \dfrac{x(x^2-1)}{(x+1)(x-6)} = \lim_{x\to -1} \dfrac{x\cancel{(x+1)}(x-1)}{\cancel{(x+1)}(x-6)}$

$= \lim_{x\to -1} \dfrac{x(x-1)}{(x-6)}$

$= \dfrac{(-1)(-2)}{(-7)}$

$= -\dfrac{2}{7}$

(b) $\lim_{x\to 2} \frac{x-2}{4x^7-64x^3}$ (Note: looks like $\frac{0}{0}$!)

$$= \lim_{x\to 2} \frac{x-2}{4x^3(x^4-16)} = \lim_{x\to 2} \frac{x-2}{4x^3(x^2-4)(x^2+4)}$$

$$= \lim_{x\to 2} \frac{\cancel{x-2}}{4x^3(x^2+4)(x+2)\cancel{(x-2)}}$$

$$= \lim_{x\to 2} \frac{1}{4x^3(x^2+4)(x+2)}$$

$$= \frac{1}{4(2)^3(4+4)(2+2)} = \frac{1}{4(8)(8)(4)}$$

$$= \frac{1}{1024}$$

(c) $\lim_{t\to 17} \frac{6-\sqrt{t+19}}{t-17}$ (Note: looks like $\frac{0}{0}$!)

$$= \lim_{t\to 17} \frac{6-\sqrt{t+19}}{t-17} \cdot \frac{6+\sqrt{t+19}}{6+\sqrt{t+19}} = \lim_{t\to 17} \frac{36-(t+19)}{(t-17)(6+\sqrt{t+19})}$$

$$= \lim_{t\to 17} \frac{17-t}{(t-17)(6+\sqrt{t+19})} = \lim_{t\to 17} \frac{-\cancel{(t-17)}}{\cancel{(t-17)}(6+\sqrt{t+19})}$$

$$= \lim_{t\to 17} \frac{-1}{6+\sqrt{t+19}} = \frac{-1}{6+\sqrt{36}} = -\frac{1}{12}$$

(d) $\lim_{x\to 0} \left[\frac{5}{x\sqrt{25+x}} - \frac{1}{x}\right]$ (Hint: Don't be afraid to use more than one trick in a single problem!)

(Note: looks like $\infty - \infty$ ... also means: "DO MORE WORK.")

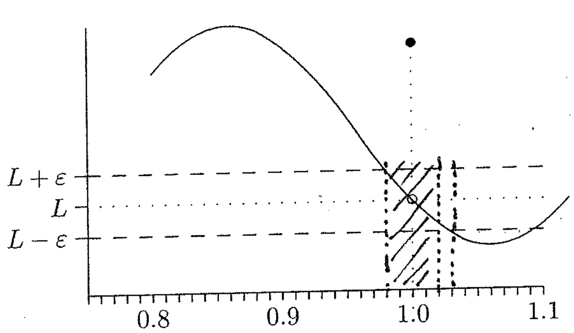
$$= \lim_{x\to 0} \left[\frac{5-\sqrt{25+x}}{x\sqrt{25+x}}\right] \cdot \frac{5+\sqrt{25+x}}{5+\sqrt{25+x}}$$

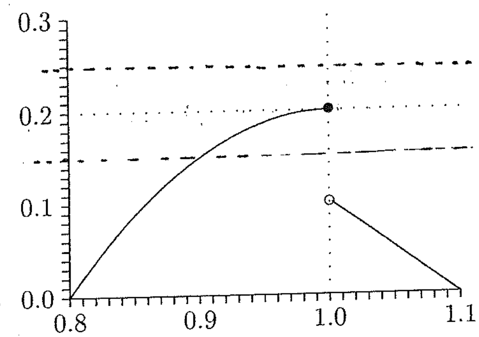
$$= \lim_{x\to 0} \left[\frac{25-(25+x)}{x(\sqrt{25+x})(5+\sqrt{25+x})}\right] = \lim_{x\to 0} \frac{-x}{x(\sqrt{25+x})(5+\sqrt{25+x})}$$

$$= \lim_{x\to 0} \frac{-1}{\sqrt{25+x}\,(5+\sqrt{25+x})} = \frac{-1}{\sqrt{25}\,(5+\sqrt{25})}$$

$$= \frac{-1}{5(10)} = -\frac{1}{50}$$

4.





(a) In the figure above (left), for the given choice of ϵ, find a δ interval about $a = 1$ which satisfies the limit definition. State the δ you have found below, and draw the appropriate vertical band in the graph above. Note scale!

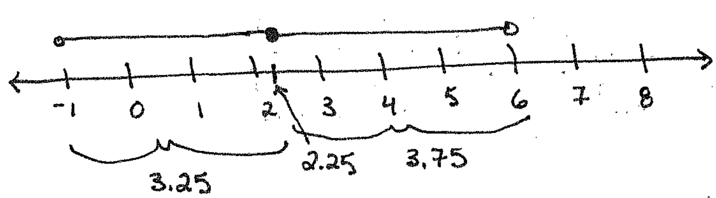
Note that the function stays in the horizontal band for x's a distance of 0.03 greater than 1, but only 0.02 less than 1. Thus we must have

$$0 < \delta \leq 0.02$$

(b) In the function on the right, $f(1) = .2$. However, show that $\lim_{x\to 1} f(x) \neq 0.2$ by finding an $\epsilon > 0$ (draw the horizontal band) for which no corresponding δ can be found. Explain why your ϵ works with a full sentence or two.

Suppose $\epsilon = 0.05$, as shown. Then there is no δ such that any x in $(1, 1+\delta)$ results in $f(x)$ in $(.2-.05, .2+.05) = (.15, .25)$

5. Suppose $|f(x) - 22| < 0.001$ whenever $-1 < x < 6$. Find all values of $\delta > 0$ such that $|f(x) - 22| < 0.001$ whenever $0 < |x - 2.25| < \delta$. (This is like a question we discussed in class on Monday!)



We are looking for the distance we are allowed to go from 2.25.

The answer is therefore $0 < \delta \leq 3.25$.

We can have an interval of length 2δ that is contained within $(-1, 6)$.

*** THIS PORTION INCLUDES ANSWERS NOT FULL SOLUTIONS ***

6. Evaluate the following limits, if they exist. If the limit does not exist, explain why. Remember good limit grammar!

   (a) $\lim\limits_{x\to 23} \dfrac{23-x}{\sqrt{x+2}-5} = -10$

   (b) $\lim\limits_{w\to 16} \dfrac{4-\sqrt{w}}{16w-w^2} = \dfrac{1}{128}$

   (c) $\lim\limits_{t\to 0} \left(\dfrac{1}{t\sqrt{1+t}} - \dfrac{1}{t}\right) = -\dfrac{1}{2}$

   (d) $\lim\limits_{x\to 1} \dfrac{\frac{1}{x^2+1}-\frac{1}{2}}{x-1} = -\dfrac{1}{2}$

   (e) $\lim\limits_{x\to 0} \dfrac{\sqrt{4-x}-2}{x^2-x} = \dfrac{1}{4}$

7. What is the value of $\lim\limits_{x\to -2}(3x+5)$? Use the ϵ-δ definition of a limit to prove your assertion.

   $\lim\limits_{x\to -2}(3x+5) = -1$; prove with $\delta = \frac{\epsilon}{3}$

8. What is the value of $\lim\limits_{x\to -1}(5x-9)$? Use the ϵ-δ definition of a limit to prove your assertion.

   $\lim\limits_{x\to -1}(5x-9) = -14$; prove with $\delta = \frac{\epsilon}{5}$

9. If $4x-9 \le f(x) \le x^2-4x+7$ for $x \ge 0$, find $\lim\limits_{x\to 4} f(x)$.

   Use the Squeeze Theorem to show $\lim\limits_{x\to 4} f(x) = 7$.

10. If $2x \le g(x) \le x^4 - x^2 + 2$ for all x, find $\lim\limits_{x\to 1} g(x)$.

    Use the Squeeze Theorem to show $\lim\limits_{x\to 1} g(x) = 2$.

11. Let $f(x) = \dfrac{x^2+3x-28}{|x+7|}$.

    (a) Write $f(x)$ as a piecewise function.

    $$f(x) = \begin{cases} \dfrac{x^2+3x-28}{-(x+7)} & \text{if } x < -7 \\ \dfrac{x^2+3x-28}{x+7} & \text{if } x > -7 \end{cases}$$

    (b) Evaluate the following limits, if they exist. If the limit does not exist, explain why.

    (i) $\lim\limits_{x\to 5} f(x) = 1$

    (ii) $\lim\limits_{x\to -7^+} f(x) = -11$

    (iii) $\lim\limits_{x\to -7} f(x)$ DNE since $\lim\limits_{x\to -7^+} f(x) \ne \lim\limits_{x\to -7^-} f(x)$.