Evaluating the Continuity of a Function

Find the interval(s) on which $k(x) = |\ln x|$ is continuous.

Solution: Let $f(x) = |x|$ and $g(x) = \ln x$. Then $k(x) = (f \circ g)(x)$.

Note that $f(x)$ is a piecewise function so we must check each of its pieces in addition to the point where the pieces join, separately for continuity.

If $x < 0$, then $|x| = -x$ is a polynomial. Hence we know f is continuous on $(-\infty, 0)$, since polynomials are continuous everywhere by Theorem 2.10.

Similarly, if $x \geq 0$, then $|x| = x$ is also a polynomial and so f is continuous on $(0, \infty)$, again by Theorem 2.10.

Next we must show that f is continuous at zero. To do this, we will check the left and right hand limits of f as x approaches zero.

$$\lim_{x \to 0^-} |x| = \lim_{x \to 0^-} -x = 0 \qquad \text{and} \qquad \lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0$$

Since these limits are equal, $\lim\limits_{x \to 0} |x| = 0 = f(0)$ and so f is also continuous at zero, and therefore continuous everywhere.

Since g is a transcendental function, by Theorem 2.15 g is continuous on its domain, which is $(0, \infty)$.

Since f is continuous everywhere, by Theorem 2.11, $k(x)$ is continuous everywhere g is continuous, that is, at all $g(a)$ where a is in $(0, \infty)$.

Thus since g is continuous on $(0, \infty)$, and f is continuous for all $g(a)$, their composition, $(f \circ g)(x)$ is also continuous on $(0, \infty)$, and thus the interval of continuity is $(0, \infty)$.