WEEK 11 LAB

MATH 131 Section 2
November 8, 2018
Covering Sections 7.1-7.8

Your Name (Print): ANSWER KEY

Your very first step for all improper integrals should be to check to see if you need to split the integral into two or more integrals and then rewrite each integral as an appropriate limit! Do not start integrating until you have done these two steps!!! Remember if you have more than one integral to evaluate, do just one at a time. Only continue with the second (third, etc.) integrals if the first is convergent.

1. Evaluate $\int_0^{\frac{\pi}{3}} \frac{\cos x}{\sqrt{\sin x}}\, dx$.

$$= \lim_{t\to 0^+} \int_t^{\pi/3} \frac{\cos x}{\sqrt{\sin x}}\, dx$$

$u = \sin x$
$du = \cos x\, dx$

If $x = t$, then $u = \sin t$.
If $x = \frac{\pi}{3}$, then $u = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

$$= \lim_{t\to 0^+} \int_{\sin t}^{\sqrt{3}/2} u^{-1/2}\, du$$

$$= \lim_{t\to 0^+} 2u^{1/2}\Big]_{\sin t}^{\sqrt{3}/2}$$

$$= \lim_{t\to 0^+} 2\left(\sqrt{\frac{\sqrt{3}}{2}} - \underbrace{\sqrt{\sin t}}_{\to 0}\right)$$

$$= 2\cdot\frac{\sqrt[4]{3}}{\sqrt{2}} - 0$$

$$= \boxed{\sqrt{2}\cdot\sqrt[4]{3}\ ,\ \text{convergent}}$$

2. Evaluate $\int_{\frac{1}{2}}^{\infty} \frac{1}{x(\ln x)^2}\,dx$.

There exists an infinite discontinuity at $x=1$.

$$= \underbrace{\int_{\frac{1}{2}}^{1} \frac{1}{x(\ln x)^2}\,dx}_{①} + \underbrace{\int_{1}^{2} \frac{1}{x(\ln x)^2}\,dx}_{②} + \underbrace{\int_{2}^{\infty} \frac{1}{x(\ln x)^2}\,dx}_{③}$$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$$① = \int_{1/2}^{1} \frac{1}{x(\ln x)^2}\,dx = \lim_{t \to 1^-} \int_{1/2}^{t} \frac{1}{x(\ln x)^2}\,dx$$

$$= \lim_{t \to 1^-} \int_{\ln(\frac{1}{2})}^{\ln t} \frac{1}{u^2}\,du$$

$$= \lim_{t \to 1^-} -\frac{1}{u}\Bigg]_{\ln(1/2)}^{\ln t}$$

$$= \lim_{t \to 1^-} \left[-\overbrace{\frac{1}{\ln t}}^{\to \infty} + \frac{1}{\ln(1/2)} \right]$$

$$= +\infty \text{ , divergent}$$

$$\boxed{\text{Thus } \int_{1/2}^{\infty} \frac{1}{x(\ln x)^2}\,dx \text{ is divergent.}}$$

3. Evaluate $\displaystyle\int_1^\infty \frac{\arctan x}{x^2}\,dx. \quad = \lim_{t\to\infty} \int_1^t \frac{\arctan x}{x^2}\,dx \quad = ☆☆$

First we will evaluate the indefinite integral:

$$\int \frac{\arctan x}{x^2}\,dx \qquad \begin{array}{ll} u = \arctan x & dv = x^{-2}dx \\ du = \dfrac{1}{1+x^2}\,dx & v = -\dfrac{1}{x} \end{array}$$

$$= -\frac{\arctan x}{x} + \int \frac{1}{x(1+x^2)}\,dx$$

$$\frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$1 = A(x^2+1) + (Bx+C)x$$

If $x = 0$, then $1 = A$

$$1 = (A+B)x^2 + Cx + A$$

$A + B = 0 \Rightarrow 1 + B = 0 \Rightarrow B = -1$

$C = 0$

$$= -\frac{\arctan x}{x} + \int\left(\frac{1}{x} + \frac{-x}{x^2+1}\right)dx$$

$u = x^2+1$

$du = 2x\,dx$

$$= -\frac{\arctan x}{x} + \ln|x| - \frac{1}{2}\ln|x^2+1| + C$$

Thus:

$$☆☆ = \lim_{t\to\infty}\left[-\frac{\arctan x}{x} + \ln|x| - \frac{1}{2}\ln|x^2+1|\right]_1^t$$

$$= \lim_{t\to\infty}\left[\left(-\frac{\overset{\frac{\pi}{2}}{\arctan t}}{t} + \ln|t| - \frac{1}{2}\ln|t^2+1|\right) - \left(-\frac{\overset{\pi/4}{\arctan 1}}{1} + \overset{0}{\cancel{\ln 1}} - \frac{1}{2}\ln 2\right)\right]$$

$$\overset{*}{=} 0 + 0 + \frac{\pi}{4} + \frac{1}{2}\ln 2 = \boxed{\frac{\pi}{4} + \frac{1}{2}\ln 2, \text{ convergent}}$$

$$* \lim_{t\to\infty}\left(\ln|t| - \frac{1}{2}\ln|t^2+1|\right) = (\infty - \infty) = \lim_{t\to\infty} \ln\left(\frac{t}{\sqrt{t^2+1}}\right) = \lim_{t\to\infty} \ln\left(\frac{\sqrt{t^2}}{\sqrt{t^2+1}}\right)$$

$$= \lim_{t\to\infty} \ln\left(\sqrt{\frac{t^2}{t^2+1}}\right) = \frac{1}{2}\ln\left(\underbrace{\lim_{t\to\infty}\frac{t^2}{t^2+1}}_{1 \text{ by rational}}\right) = \frac{1}{2}\ln 1 = 0$$

2 = 2 = degree of num & den

4. Suppose that the rate at which Exxon Mobil extracts oil from an oil reserve is given by $r(t) = r_0 e^{-kt}$, where $r_0 = 10^7$ barrels/year and $k = 0.005$. Suppose also the estimate of the total oil reserve is 2×10^9 barrels. If the extraction continues indefinitely, will the reserve be exhausted? Show your work and explain in detail your conclusion.

If the extraction continues indefinitely, the total amount of oil extracted is:

$$\int_0^\infty r_0 e^{-kt}\,dt = \lim_{b\to\infty} \int_0^b r_0 e^{-kt}\,dt$$

$u = -kt$

$du = -k\,dt$

$$= \lim_{b\to\infty} -\frac{r_0}{k}\int_0^{-kb} e^{u}\,du$$

$$= \lim_{b\to\infty} -\frac{r_0}{k} e^{u}\Big]_0^{-kb}$$

$$= \lim_{b\to\infty} -\frac{r_0}{k}\left[\cancel{e^{-kb}}^{\,0} - e^0\right]$$

$$= \frac{r_0}{k}$$

$$= \frac{10^7}{0.005} = \frac{10^7}{\frac{5}{1000}} = 2 \times 10^9 \text{ barrels}$$

This is the amount in the reserve so it is never exhausted, but the remaining amount goes to zero as $t \to \infty$.

5. Evaluate $\displaystyle\int_2^{2\sqrt{3}} \frac{1}{x^4\sqrt{x^2-3}}\,dx$.

Let $x = \sqrt{3}\sec\theta$

$dx = \sqrt{3}\sec\theta\tan\theta\,d\theta$

$$\sqrt{x^2-3} = \sqrt{3\sec^2\theta - 3} = \sqrt{3(\sec^2\theta - 1)} = \sqrt{3\tan^2\theta} = \sqrt{3}\tan\theta$$

If $x = 2$, then $2 = \sqrt{3}\sec\theta \Rightarrow \sec\theta = \frac{2}{\sqrt{3}}$

$\Rightarrow \cos\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{6}$

If $x = 2\sqrt{3}$, then $2\sqrt{3} = \sqrt{3}\sec\theta \Rightarrow \sec\theta = 2$

$\Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$

$$= \int_{\pi/6}^{\pi/3} \frac{1}{[\sqrt{3}\sec\theta]^{\cancel{4}\,3}\cdot\sqrt{3}\,\cancel{\tan\theta}} \cdot \sqrt{3}\,\cancel{\sec\theta}\,\cancel{\tan\theta}\,d\theta$$

$$= \frac{1}{9}\int_{\pi/6}^{\pi/3} \frac{1}{\sec^3\theta}\,d\theta = \frac{1}{9}\int_{\pi/6}^{\pi/3} \cos^3\theta\,d\theta$$

$$= \frac{1}{9}\int_{\pi/6}^{\pi/3} \cos^2\theta\cdot\cos\theta\,d\theta = \frac{1}{9}\int_{\pi/6}^{\pi/3} (1-\sin^2\theta)\cos\theta\,d\theta$$

Let $u = \sin\theta$, then

$du = \cos\theta\,d\theta$

If $\theta = \frac{\pi}{6}$, $u = \sin\frac{\pi}{6} = \frac{1}{2}$.

If $\theta = \frac{\pi}{3}$, $u = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

$$= \frac{1}{9}\int_{1/2}^{\sqrt{3}/2} (1-u^2)\,du$$

$$= \frac{1}{9}\left(u - \frac{u^3}{3}\right)\Bigg]_{1/2}^{\sqrt{3}/2}$$

$$= \frac{1}{9}\left[\frac{\sqrt{3}}{2} - \frac{1}{3}\left(\frac{\sqrt{3}}{2}\right)^3 - \frac{1}{2} + \frac{1}{3}\left(\frac{1}{2}\right)^3\right]$$

$$= \frac{1}{9}\left[\frac{\sqrt{3}}{2} - \frac{3\sqrt{3}}{3\cdot 8} - \frac{1}{2} + \frac{1}{3\cdot 8}\right] = \frac{1}{9}\left[\frac{12\sqrt{3} - 3\sqrt{3} - 12 + 1}{12\cdot 2}\right]$$

$$= \frac{1}{9}\left[\frac{9\sqrt{3} - 11}{24}\right] = \frac{9\sqrt{3} - 11}{216} \quad \text{or} \quad \frac{\sqrt{3}}{24} - \frac{11}{216}$$

6. Evaluate $\int_{-\infty}^{-4} \frac{1}{(x+1)^2(x+2)} dx$.

$\boxed{\text{convergent to } \frac{1}{3} + \ln\left(\frac{2}{3}\right)}$

7. Evaluate $\int_0^{\infty} \frac{1}{\sqrt{x}(1+x)} dx$. (Hint for integration: Try letting $u = \sqrt{x}$.)

$\boxed{\text{convergent to } \pi}$