WEEK 13 LAB

MATH 131 Section 2
November 29, 2018
Covering Sections 8.1-8.4

Your Name (Print): ANSWER KEY

Most of your answers for this lab should include at least one full sentence!

1. Write an expression for the general term, a_n, of the sequence. Note that in the given notation, **it is assumed that n begins at 1**. Here is a rare exception: you need not justify your answer!

(a) $\{3, 7, 11, 15, \ldots\}$

$$a_n = 4n - 1$$

(b) $\{7, 5, 7, 5, 7, 5, ...\}$

$$a_n = 6 + (-1)^{n+1}$$

2. Determine whether the following sequences are convergent or divergent. If the sequence converges, find the limit. Be sure to show work to support your answers.

(a) $\left\{ \dfrac{\ln(e^n - 4)}{8n} \right\}$

Let $f(x) = \dfrac{\ln(e^x-4)}{8x}$, for $x \in \mathbb{R}$, $x > 0$. So $f(n) = a_n$.

$$\lim_{x\to\infty} \frac{\ln(e^x-4)}{8x} = \left(\frac{\infty}{\infty}\right) \overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{e^x-4} \cdot e^x}{8} = \lim_{x\to\infty} \frac{e^x}{8(e^x-4)} = \left(\frac{\infty}{\infty}\right)$$

$$\overset{H}{=} \lim_{x\to\infty} \frac{e^x}{8e^x} = \lim_{x\to\infty} \frac{1}{8} = \frac{1}{8}$$

Therefore the sequence $\left\{ \dfrac{\ln(e^n-4)}{8n} \right\}$ converges to $\frac{1}{8}$.

(b) $\left\{\dfrac{n!}{n(n-4)!}\right\}_{n=4}^{\infty}$

$$a_n = \frac{n!}{n(n-4)!} = \frac{\cancel{n}\cdot(n-1)\cdot(n-2)\cdot(n-3)\cdot\cancel{(n-4)!}}{\cancel{n}\,\cancel{(n-4)!}} = (n-1)(n-2)(n-3)$$

So $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty}(n-1)(n-2)(n-3) = \infty.$

Thus $\left\{\dfrac{n!}{n(n-4)!}\right\}$ diverges.

(c) $\left\{\dfrac{n\sin n}{n^2+1}\right\}$

Recall that $-1 \le \sin n \le 1$ for all n and so

$$\frac{-n}{n^2+1} \le \frac{n\sin n}{n^2+1} \le \frac{n}{n^2+1} \quad \text{for all } n \ge 1.$$

Since $\displaystyle\lim_{n\to\infty}\frac{-n}{n^2+1} \underset{\text{Powers}}{\overset{\text{High}}{=}} \lim_{n\to\infty}\frac{-n}{n^2} = \lim_{n\to\infty}\frac{-1}{n} = 0,$

and $\displaystyle\lim_{n\to\infty}\frac{n}{n^2+1} \underset{\text{Powers}}{\overset{\text{High}}{=}} \lim_{n\to\infty}\frac{n}{n^2} = \lim_{n\to\infty}\frac{1}{n} = 0,$

$\displaystyle\lim_{n\to\infty}\frac{n\sin n}{n^2+1} = 0$ by the Squeeze Theorem.

Thus $\left\{\dfrac{n\sin n}{n^2+1}\right\}$ converges to zero.

3. Determine whether the following series are convergent or divergent. If a series is convergent, find the sum (if possible!). If it is divergent, explain why.

(a) $8+6+\frac{9}{2}+\frac{27}{8}+\cdots$

$$= \sum_{n=0}^{\infty} 8\left(\frac{3}{4}\right)^n$$

This is a geometric series with $a=8$ and $r=\frac{3}{4}$.

Since $-1<\frac{3}{4}=r<1$, this series converges,

and $\displaystyle\sum_{n=0}^{\infty} 8\left(\frac{3}{4}\right)^n = \frac{8}{1-\frac{3}{4}} = \frac{8}{\frac{1}{4}} = 32.$

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

Let $f(x)=\frac{1}{x\sqrt{\ln x}}$. Then f is continuous and positive for all $x \geq 2$

and $f(n)=a_n$. Now $f'(x)=\frac{d}{dx}\left(x\sqrt{\ln x}\right)^{-1} = -\left(x\sqrt{\ln x}\right)^{-2}\left(x\cdot\frac{1}{2}(\ln x)^{-\frac{1}{2}}\cdot\frac{1}{x}+\sqrt{\ln x}\right)$

So $f'(x) = \dfrac{\frac{1}{2\sqrt{\ln x}}+\sqrt{\ln x}}{-\left(x\sqrt{\ln x}\right)^2} < 0$ for all $x \geq 2$. Thus f is decreasing

for all $x \geq 2$.

$$\int_2^{\infty}\frac{1}{x\sqrt{\ln x}}\,dx = \lim_{t\to\infty}\int_2^t \frac{1}{x\sqrt{\ln x}}\,dx \quad \begin{array}{l} u=\ln x \\ du=\frac{1}{x}dx \end{array} \quad = \lim_{t\to\infty}\int_{\ln 2}^{\ln t} u^{-1/2}\,du$$

$$= \lim_{t\to\infty} 2u^{1/2}\Big]_{\ln 2}^{\ln t} = \lim_{t\to\infty} 2\left[\overset{\infty}{\cancel{\sqrt{\ln t}}} - \sqrt{\ln 2}\right] = \infty$$

Hence $\displaystyle\int_2^{\infty}\frac{1}{x\sqrt{\ln x}}\,dx$ is divergent and therefore

$\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\sqrt{\ln n}}$ is divergent by the Integral Test.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{6}{n^2+3n} = \sum_{n=1}^{6} \frac{6}{n(n+3)} = 2\sum_{n=1}^{6} \left(\frac{1}{n} - \frac{1}{n+3}\right)$

$$\frac{6}{n(n+3)} = \frac{A}{n} + \frac{B}{n+3}$$

$$6 = A(n+3) + Bn$$

If $n=0$, then $6 = 3A$, so $A=2$.

If $n=-3$, then $6=-3B$, so $B=-2$.

$$S_n = 2\Big[\left(1-\cancel{\tfrac{1}{4}}\right) + \left(\tfrac{1}{2} - \cancel{\tfrac{1}{5}}\right) + \left(\tfrac{1}{3} - \cancel{\tfrac{1}{6}}\right) + \left(\cancel{\tfrac{1}{4}} - \cancel{\tfrac{1}{7}}\right) + \left(\cancel{\tfrac{1}{5}} - \cancel{\tfrac{1}{8}}\right) + \left(\cancel{\tfrac{1}{6}} - \cancel{\tfrac{1}{9}}\right)$$
$$+ \cdots + \left\{\left(\cancel{\tfrac{1}{n-2}}\right) - \left(\tfrac{1}{n+1}\right)\right\} + \left\{\left(\cancel{\tfrac{1}{n-1}}\right) - \left(\tfrac{1}{n+2}\right)\right\} + \left\{\left(\cancel{\tfrac{1}{n}}\right) - \left(\tfrac{1}{n+3}\right)\right\}\Big]$$

$$= 2\left[1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}\right]$$

$$\text{So } \lim_{n\to\infty} S_n = \lim_{n\to\infty} 2\left[1 + \frac{1}{2} + \frac{1}{3} - \cancelto{0}{\frac{1}{n+1}} - \cancelto{0}{\frac{1}{n+2}} - \cancelto{0}{\frac{1}{n+3}}\right]$$

$$= 2\left[1 + \frac{1}{2} + \frac{1}{3}\right] = 2\left[\frac{6+3+2}{6}\right] = \frac{11}{3}$$

Thus the telescoping series $\displaystyle\sum_{n=1}^{\infty} \frac{6}{n^2+3n}$ converges to $\frac{11}{3}$.

(d) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{6}{n^2+3n} + \frac{1}{9^n}\right) = \underbrace{\sum_{n=1}^{\infty} \frac{6}{n^2+3n}}_{①} + \underbrace{\sum_{n=1}^{\infty} \frac{1}{9^n}}_{②}$

① is the series above which converges to $\frac{11}{3}$ as shown.

② is a geometric series with $a = \frac{1}{9}$ and $r = \frac{1}{9}$. Since $|r|<1$,
it is convergent to $\displaystyle\frac{a}{1-r} = \frac{\frac{1}{9}}{1-\frac{1}{9}} = \frac{\frac{1}{9}}{\frac{8}{9}} = \frac{1}{8}$.

Thus $\displaystyle\sum_{n=1}^{\infty} \left(\frac{6}{n^2+3n} + \frac{1}{9^n}\right)$ is convergent to $\displaystyle\frac{11}{3} + \frac{1}{8} = \frac{88+3}{24} = \frac{91}{24}$.

(e) $\sum_{n=1}^{\infty} ne^{-n^2}$

Let $f(x) = xe^{-x^2}$.

Then f is continuous and positive for all $x \geq 1$.

$f'(x) = x \cdot e^{-x^2} \cdot (-2x) + e^{-x^2}(1) = e^{-x^2}(1-2x^2)$.

Since $e^{-x^2} > 0$ for all x, $f'(x) < 0$ for all $x \geq 1$.

Thus f is decreasing for all $x \geq 1$ (or more exactly for all $x > \frac{1}{\sqrt{2}}$).

$$\int_1^{\infty} xe^{-x^2}\,dx = \lim_{t\to\infty} \int_1^t xe^{-x^2}\,dx$$

$u = -x^2$

$du = -2x\,dx$

$-\frac{1}{2}du = x\,dx$

$$= \lim_{t\to\infty} \int_{-1}^{-t^2} -\frac{1}{2}e^u\,du$$

$$= \lim_{t\to\infty} -\frac{1}{2}e^u\Big]_{-1}^{-t^2}$$

$$= \lim_{t\to\infty} \left[-\frac{1}{2}\cancel{e^{-t^2}}^{\to 0} + \frac{1}{2}e^{-1}\right]$$

$$= \frac{1}{2e}$$

Hence $\int_1^{\infty} xe^{-x^2}\,dx$ is convergent and therefore

$\sum_{n=1}^{\infty} ne^{-n^2}$ is convergent by the Integral Test.

4. Express $1.53\overline{42}$ as a geometric series, and write its sum as the ratio of two integers. Be sure to show all your work, including full sentences explaining why you may proceed with each step.

$$1.53\overline{42} = \frac{5063}{3300}$$

5. Find the values of x for which the series converges. Find the sum of the series for those values of x. Be sure to show and explain your work.

(a) $\sum_{n=0}^{\infty} 4^n x^n$

$$-\frac{1}{4} < x < \frac{1}{4} \quad \text{with sum} = \frac{1}{1-4x}$$

(b) $\sum_{n=1}^{\infty} (x-4)^n$

$$3 < x < 5, \quad \text{with sum} = \frac{x-4}{5-x}$$

6. Write an expression for the general term of the sequence: $-\frac{1}{2}, \frac{1}{3}, -\frac{2}{9}, \frac{4}{27}, -\frac{8}{81}, \ldots$. Note that in the given notation, it is assumed that n begins at 1.

$$a_n = \frac{(-1)^n 2^{n-2}}{3^{n-1}}$$

7. **Application: Medication.** Jane Riemann takes a maintenance medication: 64 mg once every 12 hrs. Every 12 hrs three-fourths of the drug is **eliminated** from her blood stream.

(a) Find a recurrence relation for the sequence $\{d_n\}_{n=1}^{\infty}$ where d_n is the amount of the drug in Dr. Riemann's bloodstream immediately after dose n.

$d_1 = 64$

$d_{n+1} = 64 + \frac{1}{4} d_n$

(b) Write out the first four terms of the sequence. Does the sequence appear to be monotonic? Explain.

64, 80, 84, 85, ... ; yes! ...

(c) Eventually the amount of the medication in Dr. Riemann's blood stream levels off. That is, the sequence has a limit. Find the limit L of the sequence. (Hint: Recall that if $\lim_{n\to\infty} d_{n+1} = L$, then $\lim_{n\to\infty} d_n = L$; we used this fact in a proof recently!) Is this a valuable piece of information for doctor's to have? Why or why not?

$L = 85\frac{1}{3}$

YES! ...

8. (a) Determine the convergence or divergence of $\left\{\frac{7^n+5}{7^{n+2}}\right\}$. If the sequence is convergent, find to what it converges. Be sure to carefully justify your work.

The sequence converges to $\frac{1}{49}$.

(b) Determine if the series $\sum_{n=1}^{\infty} \frac{7^n+5}{7^{n+2}}$ converges of diverges. If it converges, find its sum. Be sure to carefully justify your work.

The series diverges by the Test for Divergence.