WEEK 14 LAB

MATH 131 Sections 2 and 3
December 3, 2015
Due December 7, 2015 at the beginning of class
Covering Sections 8.1-8.5

Your Name (Print): ANSWER KEY

Group Member 1: ______________

Group Member 2: ______________

Group Member 3: ______________

Work these problems on separate paper first and use this as the final copy. I will collect one paper from each group. **YOU MUST SHOW ALL WORK TO RECEIVE CREDIT.** Simplify your answers so that you have gathered all like terms, cancelled where possible, and so that there are no negative exponents or fractions within fractions in your final answer. Neatness is expected!

Most of your answers for this lab will include at least one full sentence!

1. Write an expression for the general term, a_n, of the sequence. Note that in the given notation, **it is assumed that n begins at 1.** Here is a rare exception: you need not justify your answer!

(a) $\{3, 7, 11, 15, \ldots\}$

$$a_n = 4n - 1$$

(b) $\{7, 5, 7, 5, 7, 5, \ldots\}$

$$a_n = 6 + (-1)^{n+1}$$

2. Determine whether the following sequences are convergent or divergent. If the sequence converges, find the limit. Be sure to show work to support your answers.

(a) $\left\{ \dfrac{\ln(e^n - 4)}{8n} \right\}$

Let $f(x) = \dfrac{\ln(e^x - 4)}{8x}$, for $x \in \mathbb{R}$, $x > 0$. So $f(n) = a_n$.

$$\lim_{x\to\infty} \frac{\ln(e^x-4)}{8x} = \left(\frac{\infty}{\infty}\right) \overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{e^x-4} \cdot e^x}{8} = \lim_{x\to\infty} \frac{e^x}{8(e^x-4)}$$

$$= \left(\frac{\infty}{\infty}\right) \overset{H}{=} \lim_{x\to\infty} \frac{e^x}{8e^x} = \lim_{x\to\infty} \frac{1}{8} = \frac{1}{8}$$

Therefore $\left\{ \dfrac{\ln(e^n-4)}{8n} \right\}$ converges to $\dfrac{1}{8}$.

(b) $\left\{ \dfrac{n!}{n(n-4)!} \right\}_{n=4}^{\infty}$

$$a_n = \frac{n!}{n(n-4)!} = \frac{\cancel{n} \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot \cancel{(n-4)!}}{\cancel{n}\, \cancel{(n-4)!}} = (n-1)(n-2)(n-3)$$

So $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} (n-1)(n-2)(n-3) = \infty.$

Thus $\left\{ \dfrac{n!}{n(n-4)!} \right\}$ diverges.

(c) $\left\{ \dfrac{n \sin n}{n^2+1} \right\}$

Recall that $-1 \le \sin n \le 1$ for all n and so

$$\frac{-n}{n^2+1} \le \frac{n \sin n}{n^2+1} \le \frac{n}{n^2+1} \quad \text{for all } n \ge 1.$$

Since $\displaystyle\lim_{n\to\infty} \frac{-n}{n^2+1} \overset{\text{High Powers}}{=} \lim_{n\to\infty} \frac{-n}{n^2} = \lim_{n\to\infty} \frac{-1}{n} = 0,$

and $\displaystyle\lim_{n\to\infty} \frac{n}{n^2+1} \overset{\text{High Powers}}{=} \lim_{n\to\infty} \frac{n}{n^2} = \lim_{n\to\infty} \frac{1}{n} = 0,$

$\displaystyle\lim_{n\to\infty} \frac{n \sin n}{n^2+1} = 0$ by the Squeeze Theorem.

Thus $\left\{ \dfrac{n \sin n}{n^2+1} \right\}$ converges to zero.

3. Determine whether the following series are convergent or divergent. If a series is convergent, find the sum (if possible!). If it is divergent, explain why.

(a) $8+6+\frac{9}{2}+\frac{27}{8}+\cdots$

$$= \sum_{n=0}^{\infty} 8\left(\frac{3}{4}\right)^n$$

This is a geometric series with $a=8$ and $r=\frac{3}{4}$.

Since $-1<\frac{3}{4}<1$, this series converges,

and $\displaystyle\sum_{n=0}^{\infty} 8\left(\tfrac{3}{4}\right)^n = \frac{8}{1-\frac{3}{4}} = \frac{8}{\frac{1}{4}} = 32.$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^4}{(4n)!}$

Note that $a_n = \dfrac{(n!)^4}{(4n)!} > 0$ for all n.

$$\lim_{n\to\infty} \frac{\frac{[(n+1)!]^4}{(4n+4)!}}{\frac{(n!)^4}{(4n)!}} = \lim_{n\to\infty} \frac{(n+1)^4\cancel{(n!)^4}}{(4n+4)!}\cdot\frac{(4n)!}{\cancel{(n!)^4}} = \lim_{n\to\infty} \frac{(n+1)^4\,\cancel{(4n)!}}{(4n+4)(4n+3)(4n+2)(4n+1)\cancel{(4n)!}}$$

$$= \lim_{n\to\infty} \frac{(n+1)^4}{(4n+3)(4n+4)(4n+2)(4n+1)}$$

$$\overset{\text{High}}{\underset{\text{Powers}}{=}} \lim_{n\to\infty} \frac{n^4}{256n^4} = \lim_{n\to\infty}\frac{1}{256} = \frac{1}{256} < 1$$

Thus by the Ratio Test, this series is convergent.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{6}{n^2+3n} = \sum_{n=1}^{6} \frac{6}{n(n+3)} = 2\sum_{n=1}^{6}\left(\frac{1}{n} - \frac{1}{n+3}\right)$

$$\frac{6}{n(n+3)} = \frac{A}{n} + \frac{B}{n+3}$$

$$6 = A(n+3) + Bn$$

If $n=0$, then $6 = 3A$, so $A = 2$.

If $n=-3$, then $6 = -3B$, so $B = -2$.

$$S_n = 2\Big[\left(1-\cancel{\tfrac{1}{4}}\right) + \left(\tfrac{1}{2} - \cancel{\tfrac{1}{5}}\right) + \left(\tfrac{1}{3} - \cancel{\tfrac{1}{6}}\right) + \left(\cancel{\tfrac{1}{4}} - \cancel{\tfrac{1}{7}}\right) + \left(\cancel{\tfrac{1}{5}} - \cancel{\tfrac{1}{8}}\right) + \left(\cancel{\tfrac{1}{6}} - \cancel{\tfrac{1}{9}}\right)$$
$$+ \cdots + \left\{\left(\cancel{\tfrac{1}{n-2}}\right) - \left(\tfrac{1}{n+1}\right)\right\} + \left\{\left(\cancel{\tfrac{1}{n-1}}\right) - \left(\tfrac{1}{n+2}\right)\right\} + \left\{\left(\cancel{\tfrac{1}{n}}\right) - \left(\tfrac{1}{n+3}\right)\right\}\Big]$$

$$= 2\left[1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}\right]$$

$$\text{So } \lim_{n\to\infty} S_n = \lim_{n\to\infty} 2\left[1 + \frac{1}{2} + \frac{1}{3} - \cancel{\frac{1}{n+1}}^{\,0} - \cancel{\frac{1}{n+2}}^{\,0} - \cancel{\frac{1}{n+3}}^{\,0}\right]$$

$$= 2\left[1 + \frac{1}{2} + \frac{1}{3}\right] = 2\left[\frac{6+3+2}{6}\right] = \frac{11}{3}$$

Thus the telescoping series $\displaystyle\sum_{n=1}^{\infty} \frac{6}{n^2+3n}$ converges to $\frac{11}{3}$.

(d) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{6}{n^2+3n} + \frac{1}{9^n}\right) = \underset{①}{\sum_{n=1}^{\infty} \frac{6}{n^2+3n}} + \underset{②}{\sum_{n=1}^{\infty} \frac{1}{9^n}}$

① is the series above which converges to $\frac{11}{3}$ as shown.

② is a geometric series with $a = \frac{1}{9}$ and $r = \frac{1}{9}$. Since $|r| < 1$, it is convergent to $\dfrac{a}{1-r} = \dfrac{\frac{1}{9}}{1-\frac{1}{9}} = \dfrac{\frac{1}{9}}{\frac{8}{9}} = \dfrac{1}{8}$.

Thus $\displaystyle\sum_{n=1}^{\infty}\left(\frac{6}{n^2+3n} + \frac{1}{9^n}\right)$ is convergent to $\frac{11}{3} + \frac{1}{8} = \frac{88+3}{24} = \frac{91}{24}$.

(e) $\sum_{n=1}^{\infty} ne^{-n^2}$

Let $f(x) = xe^{-x^2}$.

Then f is continuous and positive for all $x \geq 1$.

$f'(x) = x \cdot e^{-x^2} \cdot (-2x) + e^{-x^2}(1) = e^{-x^2}(1-2x^2)$.

Since $e^{-x^2} > 0$ for all x, $f'(x) < 0$ for all $x \geq 1$.

Thus f is decreasing for all $x \geq 1$ (or more exactly for all $x > \frac{1}{\sqrt{2}}$).

$$\int_1^{\infty} xe^{-x^2}\,dx = \lim_{t\to\infty} \int_1^t xe^{-x^2}\,dx$$

$u = -x^2$

$du = -2x\,dx$

$-\frac{1}{2}du = x\,dx$

$$= \lim_{t\to\infty} \int_{-1}^{-t^2} -\frac{1}{2}e^u\,du$$

$$= \lim_{t\to\infty} -\frac{1}{2}e^u \Big]_{-1}^{-t^2}$$

$$= \lim_{t\to\infty} \left[-\frac{1}{2}\cancel{e^{-t^2}}^{\to 0} + \frac{1}{2}e^{-1}\right]$$

$$= \frac{1}{2e}$$

Hence $\int_1^{\infty} xe^{-x^2}\,dx$ is convergent and therefore

$\sum_{n=1}^{\infty} ne^{-n^2}$ is convergent by the Integral Test.

WEEK 14 LAB EXTRA FUN

MATH 131: Calculus II Sections 2 & 3 December 3, 2015

1. Express $1.53\overline{42}$ as a geometric series, and write its sum as the ratio of two integers. Be sure to show all your work, including full sentences explaining why you may proceed with each step.

$$\frac{5063}{3300}$$

2. Find the values of x for which the series converges. Find the sum of the series for those values of x. Be sure to show your work.

(a) $\sum_{n=0}^{\infty} 4^n x^n$

$-\frac{1}{4} < x < \frac{1}{4}$, sum $= \frac{1}{1-4x}$

(b) $\sum_{n=1}^{\infty} (x-4)^n$

$3 < x < 5$, sum $= \frac{x-4}{5-x}$

3. Write an expression for the general term of the sequence: $-\frac{1}{2}, \frac{1}{3}, -\frac{2}{9}, \frac{4}{27}, -\frac{8}{81}, \ldots$. Note that in the given notation, it is assumed that n begins at 1.

$$a_n = \frac{(-1)^n 2^{n-2}}{3^{n-1}}$$

4. **Application: Medication.** Jane Riemann takes a maintenance medication: 64 mg once every 12 hrs. Every 12 hrs three-fourths of the drug is **eliminated** from her blood stream.

(a) Find a recurrence relation for the sequence $\{d_n\}_{n=1}^{\infty}$ where d_n is the amount of the drug in Dr. Riemann's bloodstream immediately after dose n.

$d_1 = 64$

$d_{n+1} = 64 + \frac{1}{4} d_n$

(b) Write out the first four terms of the sequence. Does the sequence appear to be monotonic? Explain.

64, 80, 84, 85 ... yes ...

(c) Eventually the amount of the medication in Dr. Riemann's blood stream levels off. That is, the sequence has a limit. Find the limit L of the sequence. (Hint: Recall that if $\lim_{n\to\infty} d_{n+1} = L$, then $\lim_{n\to\infty} d_n = L$; we used this fact in a proof recently!) Is this a valuable piece of information for doctor's to have? Why or why not?

$L = 85\frac{1}{3}$

YES! ...

5. (a) Determine the convergence or divergence of $\left\{\frac{7^n+5}{7^{n+2}}\right\}$. If the sequence is convergent, find to what it converges. Be sure to carefully justify your work.

converges to $\frac{1}{49}$

(b) Determine if the series $\sum_{n=1}^{\infty} \frac{7^n+5}{7^{n+2}}$ converges of diverges. If it converges, find its sum. Be sure to carefully justify your work.

Diverges by the Test for Divergence