HOMEWORK ASSIGNMENT 1
MATH 3001 — FALL 2014

Exercises: 1.2.1 – 1.2.3.

1.2.1 (a) Prove that $\sqrt{3}$ is irrational. Does the same argument show that $\sqrt{6}$ is irrational?

SOLUTION: We proceed by contradiction. Suppose that $\sqrt{3}$ is rational. Then there exist relatively prime integers p and q for which $\sqrt{3} = p/q$. (Relatively prime means that $\gcd(p, q) = 1$, that is, p and q have no common factors.) It follows that $(p/q)^2 = 3$, and so

$$p^2 = 3q^2. \tag{1}$$

Hence 3 divides p^2, which implies that 3 divides p. (This is known as Euclid's Lemma, and to some, the Fundamental Theorem of Arithmetic.) Thus there is an integer r for which $3r = p$, and substituting back into Equation (1), we obtain

$$3r^2 = q^2. \tag{2}$$

Thus 3 divides q, and we have our contradiction. (We contradicted the fact that p and q have no common divisors.)

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The same style of argument may be used to prove that $\sqrt{6}$ is irrational, though the heart of the argument needs some additional justification.

Suppose that $\sqrt{6} = p/q$, where p and q are relatively prime integers. Then

$$p^2 = 6q^2, \tag{3}$$

which implies that 6 divides p^2. We would now like to say that 6 divides p, but this requires proof. One way to make the argument is as follows: since p^2 is divisible by 6, then

- p^2 is divisible by 2, hence 2 divides p;
- p^2 is divisible by 3, hence 3 divides p.

Both of these claims follow from Euclid's Lemma. Now, since both 2 and 3 divide p, it follows that p is divisible by 6. Thus there exists an integer r for which $6r = p$. Hence

$$6r^2 = q^2. \tag{4}$$

By the same argument, 6 divides q, and we have a contradiction.

(b) Where does the proof of Theorem 1.1.1 break down if we try to use it to prove that $\sqrt{4}$ is irrational?

SOLUTION: The crux to each of these proofs is the statement, "if a divides p^2, then a divides p". That is where this argument will fail. For the sake of seeing the argument out, here it is:

Suppose, by contradiction, that $\sqrt{4} = p/q$ is rational with p and q relatively prime. Then as before, we have

$$p^2 = 4q^2. \tag{5}$$

But now, if 4 divides p^2, we can only conclude that p is even. That is, p is divisible by 2. So $p = 2r$ for some integer r. Hence

$$r^2 = q^2, \tag{6}$$

which implies that $r = \pm q$. Thus $p = \pm 2q$. You might think this is a contradiction since now p and q share all the factors of q. But, this is not a contradiction because p and

q do not have any common factors when $q = \pm 1$. In fact, our algebraic argument has given us the solutions, which means we should not have used "proof by contradiction" in the first place.

1.2.2 Decide which of the following represent true statements about the nature of sets. For any that are false, provide a specific example where the statement in question does not hold.

(a) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \cdots$ are all sets containing an infinite number of elements, then the intersection $\cap_{n=1}^{\infty} A_n$ is infinite as well.

SOLUTION: False. See example in book, or $A_n = [-1/2^n, 1/2^n]$.

(b) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \cdots$ are all finite, nonempty sets of real numbers, then the intersection $\cap_{n=1}^{\infty} A_n$ is finite and nonempty.

SOLUTION: True.

(c) $A \cap (B \cup C) = (A \cap B) \cup C$.

SOLUTION: False. Let $A = B$, $A \neq C$, and all sets nonempty. Then $A \cap (B \cup C) = A$ and $(A \cap B) \cup C = A \cup C$.

(d) $A \cap (B \cap C) = (A \cap B) \cap C$.

SOLUTION: True.

(e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

SOLUTION: True.

1.2.3 (De Morgan's Laws.) Let A and B be subsets of $\mathbf{R}$.

(a) If $x \in (A \cap B)^c$, explain why $x \in A^c \cup B^c$. This shows that $(A \cap B)^c \subseteq A^c \cup B^c$.

SOLUTION: If $x \in (A \cap B)^c$, then $x \notin A \cap B$, which is the same as saying "not ($x \in A$ and $x \in B$)". Thus $x \notin A$ or $x \notin B$. Hence $x \in A^c$ or $x \in B^c$, i.e. $x \in A^c \cup B^c$.

(b) Prove the reverse inclusion $(A \cap B)^c \supseteq A^c \cup B^c$, and conclude that $(A \cap B)^c = A^c \cup B^c$.

SOLUTION: Suppose that $x \in A^c \cup B^c$. Then $x \in A^c$ or $x \in B^c$, so by definition $x \notin A$ or $x \notin B$. Equivalently, this is "not ($x \in A$ and $x \in B$)". Hence $x \notin A \cap B$, i.e. $x \in (A \cap B)^c$. By parts (a) and (b), we have equality of sets.

(c) Show $(A \cup B)^c = A^c \cap B^c$ by demonstrating inclusion both ways.

SOLUTION:

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\begin{aligned}
x \in (A \cup B)^c &\Leftrightarrow \neg(x \in A \cup B) \\
&\Leftrightarrow \neg(x \in A \vee x \in B) \\
&\Leftrightarrow x \notin A \wedge x \notin B \\
&\Leftrightarrow x \in A^c \wedge x \in B^c \\
&\Leftrightarrow x \in A^c \cap B^c.
\end{aligned}
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