HOMEWORK ASSIGNMENT 12
MATH 3001 — FALL 2014
DUE FRIDAY, DECEMBER 5

**Exercises:**

(1) (Exercise 7.2.1) Prove that if $f$ is a bounded function on $[a, b]$, then $L(f) \leq U(f)$.

(2) Let $f(x) = x^2$ and let $P_n = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\}$ be a partition of $[0, 1]$.

(a) Use the fact that $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ to evaluate $U(f, P_n)$ and $L(f, P_n)$.

(b) Use your expressions from part (a) to prove that $f(x)$ is Riemann-integrable on $[0, 1]$.

(c) Use your expressions from part (a) to evaluate $\int_0^1 f(x)\,dx$. (Do not use the fundamental theorem of calculus.)

(3) Let

$$f(x) = \begin{cases} 0 & \text{if } x = 2^{-n} \text{ for some } n \in \mathbf{N} \\ 1 & \text{otherwise.} \end{cases}$$

Prove that $f$ is integrable on $[0, 1]$ and compute $\int_0^1 f(x)\,dx$.

(4) (Exercise 7.5.7) If $g$ is continuous on $[a, b]$, show that there exists a point $c \in (a, b)$ such that

$$g(c) = \frac{1}{b-a} \int_a^b g(x)\,dx.$$