Suggested problems
Math 3001 — Fall 2014

Exercises:

(1) Provide all the details of the proof of Theorem 6.4.2:

> **Theorem 6.4.2.** *Let f_n be continuous functions defined on a set $A \subseteq \mathbf{R}$, and assume $\sum_{n=1}^{\infty} f_n$ converges uniformly on A to a function f. Then, f is continuous on A.*

6.3.2 Consider the sequence of functions defined by

$$g_n(x) = \frac{x^n}{n}.$$

(a) Show that (g_n) converges uniformly on $[0,1]$ and find $g = \lim g_n$. Show that g is differentiable and compute $g'(x)$ for all $x \in [0,1]$.

(b) Now, show that (g'_n) converges on $[0,1]$. Is the convergence uniform? Set $h = \lim g'_n$ and compare h to g'. Are they the same?

6.4.1 Theorem 6.4.4 in the book states:

> **Theorem 6.4.4** (Cauchy Criterion for Uniform Convergence of Series)**.** *A series $\sum_{n=1}^{\infty} f_n$ converges uniformly on $A \subseteq \mathbf{R}$ if and only if for every $\varepsilon > 0$ there exists an $N \in \mathbf{N}$ such that for all $n > m \geq N$,*
>
> $$|f_{m+1}(x) + f_{m+2}(x) + \cdots + f_n(x)| < \varepsilon$$
>
> *for all $x \in A$.*

Use Theorem 6.4.4 to prove that if $\sum_{n=1}^{\infty} g_n$ converges uniformly, then (g_n) converges uniformly to zero.

6.4.6 (sort of) Let $f_n(x) = x^n/n$.

(a) Prove that $f(x) = \sum_{n=1}^{\infty} f_n(x)$ converges pointwise on $(-1,1)$.

(b) [Hard.] Prove that for each $c \in (0,1)$, $f(x) = \sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $[-c,c]$. Conclude that f is continuous on $[-c,c]$.

(c) Prove that $\sum_{n=1}^{\infty} f_n(x)$ does not converge when $x = 1$.

(d) Does $\sum_{n=1}^{\infty} f_n(x)$ converges when $x = -1$? Prove your claim.