Homework Assignment 5
Math 3001 — Fall 2014
Due Friday, September 26

Exercises:

2.4.2. (a) Prove that the sequence defined by $x_1 = 3$ and $x_{n+1} = \dfrac{1}{4 - x_n}$ converges.

(b) Now that we know $\lim x_n$ exists, explain why $\lim x_{n+1}$ must also exist and equal the same value.

(c) Take the limit of each side of the recursive equation in part (a) of this exercise to explicitly compute $\lim x_n$.

2.4.5. Let $x_1 = 2$, and define $x_{n+1} = \dfrac{1}{2}\left(x_n + \dfrac{2}{x_n}\right)$.

(a) Show that x_n^2 is always greater than 2, and then use this to prove that $x_n - x_{n+1} \geq 0$. Conclude that $\lim x_n = \sqrt{2}$.

Remark: This problem is a special application of Newton's Method, which is an efficient algorithm for computing the roots of real-valued functions. (That is, the algorithm finds solutions to $f(x) = 0$.) In this problem, the algorithm is being applied to find a root of $x^2 - 2$.

2.4.6. (Limit Superior). Let (a_n) be a bounded sequence.

(a) Prove that the sequence defined by $y_n = \sup\{a_k \colon k \geq n\}$ converges.

(b) The *limit superior* of (a_n), or $\limsup a_n$, is defined by

$$\limsup a_n = \lim y_n,$$

where y_n is the sequence from part (a) of this exercise. Provide a reasonable definition for $\liminf a_n$ and briefly explain why it always exists for any bounded sequence.

(c) Prove that $\liminf a_n \leq \limsup a_n$ for every bounded sequence, and give an example of a sequence for which the inequality is strict.

(d) Show that $\liminf a_n = \limsup a_n$ if and only if $\lim a_n$ exists. In this case, all three share the same value.

2.5.1. Prove Theorem 2.5.2: Subsequences of a convergent sequence converge to the same limit as the original sequence.

2.5.4. Assume (a_n) is a bounded sequence with the property that every convergent subsequence of (a_n) converges to the same limit $a \in \mathbf{R}$. Show that $\lim a_n = a$.

2.6.2. Prove Theorem 2.6.2: Every convergent sequence is a Cauchy sequence.

2.6.4. Assume (a_n) and (b_n) are Cauchy sequences. Use a triangle inequality argument to prove $c_n = |a_n - b_n|$ is Cauchy.

Suggested problems: 2.5.3, 2.6.1 (You do not need to hand these in.)