Homework Assignment 6
Math 3001 — Fall 2014
Due Friday, October 3

Exercises:

2.6.3. (a) Explain how the following pseudo-Cauchy property differs from the proper definition of a Cauchy sequence: A sequence (s_n) is *pseudo-Cauchy* if, for all $\epsilon > 0$, there exists an N such that if $n \geq N$, then $|s_{n+1} - s_n| < \epsilon$.
(b) If possible, give an example of a divergent sequence (s_n) that is pseudo-Cauchy.

2.7.1. Proving the Alternating Series Test (Theorem 2.7.7) amounts to showing that the sequence of partial sums

$$s_n = a_1 - a_2 + a_3 - \cdots \pm a_n$$

converges. (The opening example in Section 2.1 includes a typical illustration of (s_n).)
(a) Prove the Alternating Series Test by showing that (s_n) is a Cauchy sequence.

2.7.3. Let $\sum a_n$ be given. For each $n \in \mathbf{N}$, let $p_n = a_n$ if a_n is positive and assign $p_n = 0$ if $a_n \leq 0$. In a similar manner, let $q_n = a_n$ if a_n is negative and $q_n = 0$ if $a_n \geq 0$.
(a) Argue that if $\sum a_n$ diverges, then at least one of $\sum p_n$ or $\sum q_n$ diverges.
(b) Show that if $\sum a_n$ converges conditionally, then both $\sum p_n$ and $\sum q_n$ diverge.

2.7.5. (a) Show that if $\sum a_n$ converges absolutely, then $\sum a_n^2$ converges absolutely. Does this proposition hold without absolute convergence?

2.7.8. Prove Theorem 2.7.1 part (ii): If $\sum a_n = A$ and $\sum b_n = B$, then $\sum(a_n + b_n) = A + B$.

2.7.9. (Ratio Test). Given a series $\sum_{n=1}^{\infty} a_n$ with $a_n \neq 0$, the Ratio Test states that if (a_n) satisfies

$$\lim \left| \frac{a_{n+1}}{a_n} \right| = r < 1,$$

then the series converges absolutely.
(a) Let r' satisfy $r < r' < 1$. (Why must such an r' exist?) Explain why there exists an N such that $n \geq N$ implies $|a_{n+1}| \leq |a_n| r'$.
(b) Why does $|a_N| \sum (r')^n$ necessarily converge?
(c) Now, show that $\sum |a_n|$ converges.

Suggested problems: 2.7.1(b,c), 2.7.2(a,b), 2.7.4, 2.7.5(b) (You do not need to hand these in.)