Homework Assignment 8
Math 3001 — Fall 2014
Due Friday, October 17

Exercises:

3.2.5. Let $a \in A$. Prove that a is an isolated point of A if and only if there exists an ϵ-neighborhood $V_\epsilon(a)$ such that $V_\epsilon(a) \cap A = \{a\}$.

3.2.6. Prove Theorem 3.2.8: A set $F \subseteq \mathbf{R}$ is closed if and only if every Cauchy sequence contained in F has a limit that is also an element of F.

3.2.9.
(a) If y is a limit point of $A \cup B$, show that y is either a limit point of A or a limit point of B (or both).
(b) Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

3.3.1. Show that if K is compact, then $\sup K$ and $\inf K$ both exist and are elements of K.

3.3.2. Show that if K is compact and F is closed, then $K \cap F$ is compact.

3.3.8. Follow these steps to prove the final implication in Theorem 3.3.8. Assume K satisfies (i) and (ii), and let $\{O_\lambda \colon \lambda \in \Lambda\}$ be an open cover for K. For contradiction, let's assume that no finite subcover exists. Let I_0 be a closed interval containing K, and bisect I_0 into two closed intervals A_1 and B_1.

(a) Why must either $A_1 \cap K$ or $B_1 \cap K$ (or both) have no finite sub cover consisting of sets from $\{O_\lambda \colon \lambda \in \Lambda\}$?

(b) Show that there exists a nested sequence of closed intervals $I_0 \supseteq I_1 \supseteq \cdots$ with the property that, for each n, $I_n \cap K$ cannot be finitely covered and $\lim |I_n| = 0$ ($|I_n|$ is the length of the interval). (cf. the proof of the Bolzano-Weierstrass theorem.)

(c) Show that there exists an $x \in K$ such that $x \in I_n$ for all n.

(d) Because $x \in K$, there must exist an one set O_{λ_0} from the original collection that contains x as an element. Argue that there must be an n_0 large enough to guarantee that $I_{n_0} \subseteq O_{\lambda_0}$. Explain why this furnishes us with the desired contradiction.