# Cantor set

## Math 3001 — Fall 2014

The Cantor set is a subset of the interval $[0,1]$, which is defined as follows. Let $C_0 = [0,1]$, and make the set $C_1$ by removing the open middle-third interval of $C_0$. That is,

$$C_1 = C_0 \setminus \left(\frac{1}{3}, \frac{2}{3}\right) = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right].$$

Now make $C_2$ by removing the open middle-thirds of each of the intervals of $C_1$:

$$C_2 = C_1 \setminus \left(\frac{1}{9}, \frac{2}{9}\right) \setminus \left(\frac{7}{9}, \frac{8}{9}\right) = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right].$$

Continuing this process, define $C_n$ to be the set $C_{n-1}$ with the open middle-third of each of its intervals removed. We now define the Cantor set $C$ as

$$C = \bigcap_{n=0}^{\infty} C_n.$$

(1) Draw a picture of $C_0$, $C_1$, $C_2$, $C_3$, $\ldots$ to get a feel for these sets.
(2) Prove that $C$ is non-empty. What are some of the values in the Cantor set? Are 0 and 1 in $C$? Are there other values in $C$?
(3) Prove that $C_n$ is a closed for every $n \in \mathbf{N}$.
(4) Prove that $C$ is compact.
(5) Prove that $C$ contains no isolated points.
(6) To go from $C_0$ to $C_1$, we remove $1/3$ of the interval.
    To go from $C_0$ to $C_2$, we first remove the middle $1/3$ of $[0,1]$ (getting to $C_1$), then we remove $2/9$ of the interval $[0,1]$ (that is, $1/9$ from each of the intervals of $C_1$) to get to $C_2$. So in total, we remove $\frac{1}{3} + \frac{2}{9}$ of the interval $[0,1]$ to go from $C_0$ to $C_2$.
    Between $C_0$ and $C_3$, how much of the interval is removed? From $C_0$ to $C_4$?
(7) How much of the interval is removed between $C_0$ and $C$? In other words, what is the "length" of $C$?
(8) Let $D_n$ be the set of decimals of length $n$ consisting of only 0's and 1's. For example,

$$D_1 = \{0.0, 0.1\}$$
$$D_2 = \{0.00, 0.01, 0.10, 0.11\}.$$

    Prove that the elements of $D_n$ are in a one-to-one correspondence with the intervals of $C_n$.
(9) What number in the Cantor set does $0.\overline{0}$ represent? How about $0.\overline{1}$? $0.0\overline{1}$?
(10) Let $D$ denote the set of all infinite decimals consisting of only 0's and 1's. Use the nested interval property to prove that each element in $D$ corresponds to an element of the cantor set.
(11) What is the cardinality of the Cantor set? (Is it finite? Countably infinite? Uncountable?)