Homework Assignment 3
Math 3001 — Fall 2014
Due Friday, September 12

Exercises:

1.3.2. (a) Write a formal definition in the style of Definition 1.3.2 for the *infimum* or *greatest lower bound* of a set.
(b) State and prove a version of Lemma 1.3.7 for greatest lower bounds.

1.3.3. (a) Let A be bounded below, and define $B = \{b \in \mathbf{R} \colon b \text{ is a lower bound for } A\}$. Show that $\sup B = \inf A$.

1.3.4. Assume that A and B are nonempty, bounded above, and satisfy $B \subseteq A$. Show $\sup B \leq \sup A$.

1.3.8. If $\sup A < \sup B$, then show that there exists an element $b \in B$ that is an upper bound for A.

1.3.9. Without worrying about formal proofs for the moment, decide if the following statements about supreme and infima are true or false. For any that are false, supply an example where the claim in question does not appear to hold.
(a) A finite, nonempty set always contains its supremum.
(b) If $a < L$ for every element a in the set A, then $\sup A < L$.
(c) If A and B are sets with the property that $a < b$ for every $a \in A$ and every $b \in B$, then it follows that $\sup A < \sup B$.
(d) If $\sup A = s$ and $\sup B = t$, then $\sup(A + B) = s + t$. The set $A + B$ is defined as $A + B = \{a + b \colon a \in A \text{ and } b \in B\}$.
(e) If $\sup A \leq \sup B$, then there exists an element $b \in B$ that is an upper bound for A.

1.4.2. Let $\mathbf{I}$ denote the irrational numbers.
(a) Show that if $a, b \in \mathbf{Q}$, then ab and $a + b$ are elements of $\mathbf{Q}$ as well.
(b) Show that if $a \in \mathbf{Q}$ and $t \in \mathbf{I}$, then $a + t \in \mathbf{I}$ and $at \in \mathbf{I}$ as long as $a \neq 0$.
(c) Part (a) can be summarized by saying that $\mathbf{Q}$ is closed under addition and multiplication. Is $\mathbf{I}$ closed under addition and multiplication? Given two irrational numbers s and t, what can you say about $s + t$ and st.

1.4.3. Using Exercise 1.4.2, supply a proof for Corollary 1.4.4 by applying Theorem 1.4.3 to the real numbers $a - \sqrt{2}$ and $b - \sqrt{2}$.

Read: Read the section on Countable and Uncountable sets, pages 22–26.

1.4.8. Use the following outline to supply proofs for the statements in Theorem 1.4.13.
(a) First, prove statement (i) for two countable sets, A_1 and A_2. Example 1.4.8 (ii) may be a useful reference. Some technicalities can be avoided by first replacing A_2 with the set $B_2 = A_2 \setminus A_1 = \{x \in A_2 \colon x \notin A_1\}$. The point of this is that the union $A_1 \cup B_2$ is equal to $A_1 \cup A_2$ and the sets A_1 and B_2 are disjoint. (What happens if B_2 is finite?) Now, explain how the more general statement in (i) follows.
(b) Explain why induction cannot be used to prove part (ii) of Theorem 1.4.13 from part (i).
(c) Show how arranging $\mathbf{N}$ into the two-dimensional array

$$\begin{array}{ccccc}
1 & 3 & 6 & 10 & 15 & \cdots \\
2 & 5 & 9 & 14 & \ddots \\
4 & 8 & 13 & \ddots \\
7 & 12 & \ddots \\
11 & \ddots \\
\vdots
\end{array}$$

leads to a proof of Theorem 1.4.13 (ii).