Analysis — Spring 2015
CU Boulder Math 3001

1. Warm-up Proofs

Definition 1. An integer n is *even* if $n = 2k$ for some integer k.

Definition 2. An integer n is *odd* if $n = 2k + 1$ for some integer k.

Exercise 3. Prove that the sum of two even integers is even.

Exercise 4. Prove that the sum of two odd integers is even.

Exercise 5. Prove that the product of two odd integers is odd.

Exercise 6. Prove that the product of two even integers is even.

Exercise 7. Prove that there is no integer that is both even and odd.

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Lemma 8 (Euclid's lemma)**.** *Let p be a prime number and a and b integers. If $p \mid ab$, then $p \mid a$ or $p \mid b$.*

Exercise 9. Prove that $\sqrt{2}$ is irrational.

Exercise 10. Is $\sqrt{3}$ irrational? Prove your claim.

Exercise 11. Is $\sqrt{6}$ irrational? Prove your claim.

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Exercise 12. Prove that $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

Exercise 13. Suppose $a_1 = 1$ and $a_{n+1} = 1 + \frac{1}{2a_n}$ for each $n \in \mathbb{N}$. Prove that $1 \le a_n < 2$ for all $n \in \mathbb{N}$.

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Let A and B be sets.

Exercise 14. Prove that $(A \cap B)^c = A^c \cup B^c$.

Exercise 15. Prove that $(A \cup B)^c = A^c \cap B^c$.

Exercise 16. Let $A_1, A_2, \ldots$ be sets. Prove that $\left(\displaystyle\bigcup_{k=1}^{n} A_k\right)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c$.

Proof. 1) For n=1, $\left(\displaystyle\bigcup_{k=1}^{1} A_k\right)^c = A_1^c$ is definitely true.

2) If $\left(\displaystyle\bigcup_{k=1}^{n} A_k\right)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c$ is true.

Then

$$(\bigcup_{k=1}^{n+1} A_k)^c = (A_1 \cup A_2 \cup \cdots \cup A_n \cup A_{n+1})^c \tag{1}$$

$$= \left((\bigcup_{k=1}^{n} A_k) \cup A_{n+1} \right)^c \tag{2}$$

$$= (\bigcup_{k=1}^{n} A_k)^c \cap (A_{n+1})^c \tag{3}$$

$$= A_1^c \cap A_2^c \cap \cdots \cap A_n^c \cap A_{n+1} \tag{4}$$

$\therefore$ By induction, the statement $\left(\bigcup_{k=1}^{n} A_k \right)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c$ is true. $\square$