Analysis — Spring 2015
CU Boulder Math 3001

worksheet 10

Read section: 4.2

Definition. Let $f\colon A \to \mathbb{R}$, and let c be a limit point of A. We say that $\lim_{x\to c} f(x) = L$ if, for each $\epsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x - c| < \delta$ (and $x \in A$), then $|f(x) - L| < \epsilon$.

Equivalently,

Definition. Let $f\colon A \to \mathbb{R}$, and let c be a limit point of A. We say $\lim_{x\to c} f(x) = L$ if for every $V_\epsilon(L)$, there exists $V_\delta(c)$ such that for each $x \in (V_\delta(c) \setminus \{c\})$, it follows that $f(x) \in V_\epsilon(L)$.

Exercise 1. Use the definition of functional limit (above) to prove each of the following.

(a) $\lim\limits_{x\to 2} 3x + 1 = 7$.
(b) $\lim\limits_{x\to 3} x^2 - 9 = 0$.
(c) $\lim\limits_{x\to 2} x^3 = 8$.

Exercise 2. Prove

Theorem (Sequential criterion for functional limits). Let $f\colon A \to \mathbb{R}$, and let c be a limit point of A. Then $\lim_{x\to c} f(x) = L$ if and only if for all sequences (x_n) in A satisfying $x_n \neq c$ (for all $n \in \mathbb{N}$) and $(x_n) \to c$, it follows that the sequence $(f(x_n))$ converges to L.

Exercise 3. Prove

Theorem (Algebraic limit theorem for functional limits). Let $f\colon A \to \mathbb{R}$, $g\colon A \to \mathbb{R}$, and let c be a limit point of A. Suppose that $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$. Then

(a) $\lim_{x\to c} kf(x) = kL$ for all $k \in \mathbb{R}$,
(b) $\lim_{x\to c}[f(x) + g(x)] = L + M$,
(c) $\lim_{x\to c} f(x)g(x) = LM$,
(d) $\lim_{x\to c} f(x)/g(x) = L/M$ provided that $M \neq 0$.

Exercise 4. Prove

Theorem (Divergence criterion for functional limits). Let $f\colon A \to \mathbb{R}$, and let c be a limit point of A. The limit $\lim_{x\to c} f(x)$ does not exist if there exist sequences (x_n) and (y_n) in A, where $x_n \neq c$ and $y_n \neq c$ (for all $n \in \mathbb{N}$), and

$$\lim x_n = \lim y_n = c \quad \text{but} \quad \lim f(x_n) \neq \lim f(y_n).$$

Exercise 5. Prove that $\lim_{x\to 0} |x|/x$ does not exist.

Exercise 6.

(a) Provide a rigorous definition of what it means for a function f to be bounded on a set $A \subseteq \mathbb{R}$.
(b) Let $g\colon A \to \mathbb{R}$ and assume that f is a bounded function on A. Show that if $\lim_{x\to c} g(x) = 0$, then $\lim_{x\to c} f(x)g(x) = 0$.

Exercise 7. Let $f\colon A \to \mathbb{R}$, $g\colon A \to \mathbb{R}$, and let c be a limit point of A. Assume that $f(x) \geq g(x)$ for all $x \in A$. Prove that $\lim_{x\to c} f(x) \geq \lim_{x\to c} g(x)$.

Exercise 8. Let f, g, and h be functions from A to $\mathbb{R}$ that satisfy $f(x) \leq g(x) \leq h(x)$ for all $x \in A$. Let c be a limit point of A. Prove that if $\lim_{x\to c} f(x) \leq \lim_{x\to c} h(x) = L$, then $\lim_{x\to c} g(x) = L$.