Analysis — Spring 2015
CU Boulder Math 3001

worksheet 11

Read sections: 4.3–4.4

Definition. A function $f\colon A \to \mathbb{R}$ is *continuous* at a point $c \in A$ if, for each $\epsilon > 0$, there exists $\delta > 0$ such that if $|x - c| < \delta$, it follows that $|f(x) - f(c)| < \epsilon$.

If f is continuous at every point in its domain A, then f is *continuous* on A.

Exercise 1. Prove:

Theorem (Characterizations of continuity)**.** Let $f\colon A \to \mathbb{R}$, and let $c \in A$ be a limit point of A. The function f is continuous at c if and only if at least one of the following conditions is met.

i.) For all $\epsilon > 0$, there exists $\delta > 0$ such that $|x-c| < \delta$ (and $x \in A$) implies $|f(x)-f(c)| < \epsilon$.
ii.) $\lim_{x\to c} f(x) = f(c)$.
iii.) For all $V_\epsilon(f(c))$, there exists $V_\delta(c)$ such that if $x \in V_\delta(c)$ (and $x \in A$), then $f(x) \in V_\epsilon(f(c))$.
iv.) If $(x_n) \to c$ (and $x_n \in A$), then $(f(x_n)) \to f(c)$.

Exercise 2. Prove:

Theorem (Criterion for discontinuity)**.** Let $f\colon A \to \mathbb{R}$, and let $c \in A$ be a limit point of A. If there exists a sequence $(x_n) \subseteq A$ where $(x_n) \to c$ but $(f(x_n))$ does not converge to $f(c)$, then f is not continuous at c.

Exercise 3. Prove that $f(x) = \sqrt[3]{x}$ is continuous on $[0, \infty)$. [Hint: the identity $(a^3 - b^3) = (a-b)(a^2 + ab + b^2)$ may be helpful.]

Exercise 4. Let

$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{m}{n} \in \mathbb{Q} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Prove that f is continuous at every irrational number, and discontinuous at every rational number.

Exercise 5. Prove that $f\colon \mathbb{Z} \to \mathbb{R}$ is continuous on $\mathbb{Z}$.

Exercise 6. Prove:

Theorem (Algebraic continuity theorem)**.** Assume $f\colon A \to \mathbb{R}$ and $g\colon A \to \mathbb{R}$ are continuous at a point $c \in A$. Then,

i.) $kf(x)$ is continuous at c for all $k \in \mathbb{R}$.
ii.) $f(x) + g(x)$ is continuous at c.

iii.) $f(x)g(x)$ is continuous at c.
iv.) $f(x)/g(x)$ is continuous at c, provided the quotient is defined.

Exercise 7. Let $f\colon A \to \mathbb{R}$, $g\colon B \to \mathbb{R}$, and suppose $f(A) \subseteq B$. Prove that if $f(x)$ is continuous at $c \in A$, and $g(x)$ is continuous at $f(c) \in B$, then $g(f(x))$ is continuous at c.

Exercise 8. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$. Prove that $B = \{x\colon f(x) = 0\}$ is a closed set.

Exercise 9. Let $f\colon \mathbb{R} \to \mathbb{R}$, and assume there exists $c \in (0,1)$ for which

$$|f(x) - f(y)| \leq c|x - y|$$

for all $x, y \in \mathbb{R}$.

i.) Prove that f is continuous on $\mathbb{R}$.
ii.) Fix a value $a \in \mathbb{R}$. Prove that the sequence $(a_n) = (f(a), f(f(a)), f(f(f(a))), \ldots)$ is a Cauchy sequence.
iii.) Let $a = \lim a_n$. Prove that a is the unique fixed point of $f(x)$. That is, prove that $f(a) = a$, and that $x = a$ is the only value for which $f(x) = x$.
iv.) Prove that the sequence $(x_n) = (f(x), f(f(x)), f(f(f(x))), \ldots)$ converges to a for every $x \in \mathbb{R}$.

Exercise 10. Prove:

Theorem. Suppose $f\colon A \to \mathbb{R}$ is continuous on A. If A is compact, then $f(A)$ is compact.

Exercise 11. Prove:

Theorem (Extreme value theorem)**.** If $f\colon A \to \mathbb{R}$ is continuous on a compact set A, then f attains a maximum and minimum on A. That is, there exist m and M in A such that $f(m) \leq f(x) \leq f(M)$ for all $x \in A$.

Exercise 12. Prove that each of the following statements is false.

i.) If $f\colon A \to \mathbb{R}$ and A is compact, then f attains a maximum and minimum on A.
ii.) If $f\colon A \to \mathbb{R}$ is continuous on a closed set A, then f attains a maximum and minimum on A.
iii.) If $f\colon A \to \mathbb{R}$ is continuous on a bounded set A, then f attains a maximum and minimum on A.