Analysis — Spring 2015
CU Boulder Math 3001

worksheet 12

Read sections: 4.4

Definition. A function $f\colon A \to \mathbb{R}$ is *uniformly continuous on* A if for each $\epsilon > 0$, there exists $\delta > 0$ such that if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

Exercise 1. Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$.

Exercise 2. Suppose f is defined on an open interval (a, c) and it is known that f is uniformly continuous on $(a, b]$ and $[b, c)$. Prove that f is uniformly continuous on (a, c).

Exercise 3. Prove:

Theorem (Sequential criterion for nonuniform continuity)**.** A function $f\colon A \to \mathbb{R}$ fails to be uniformly continuous on A if and only if there exists an $\epsilon > 0$ and two sequences (x_n) and (y_n) in A for which the sequence $(x_n - y_n)$ converges to 0, but $|f(x_n) - f(y_n)| \geq \epsilon$ for all n.

Exercise 4. Prove that $f(x) = 1/x^2$ is uniformly continuous on $[1, \infty)$, but is not uniformly continuous on $(0, 1]$.

Exercise 5. Prove:

Theorem. A function that is continuous on a compact set K is uniformly continuous on K.

Definition. A function $f\colon A \to \mathbb{R}$ is called *Lipshitz* if there exists a bound $M > 0$ such that

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq M$$

for all $x, y \in A$.

Exercise 6. Prove that if f is Lipschitz on A, then f is uniformly continuous on A.