Analysis — Spring 2015
CU Boulder Math 3001

worksheet 13

Read sections: 5.1–5.2

Definition. Let $f\colon A \to \mathbb{R}$ be a function defined on an interval A. The *derivative of f at $c \in A$* is

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c},$$

and we say that f is *differentiable* at c if this limit exists. If $f'(c)$ exists for all $c \in A$, then f is *differentiable on A*.

Exercise 1. Prove that $f(x) = x^n$ is differentiable on $\mathbb{R}$, and moreover, $f'(x) = nx^{n-1}$.

Exercise 2. Let $f(x) = |x|$.
a.) Prove that $f(x)$ is differentiable on $\mathbb{R} \setminus \{0\}$.
b.) Prove that $f(x)$ is not differentiable at $c = 0$.

Exercise 3. Consider the family of functions

$$g_n(x) = \begin{cases} x^n \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0, \end{cases}$$

where $n \in \mathbb{N} \cup \{0\}$.
a.) Prove that $g_n(x)$ is continuous at $c = 0$ whenever $n \geq 1$, but $g_0(x)$ is not continuous at $c = 0$.
b.) Prove that $g_n(x)$ is differentiable at $c = 0$ whenever $n \geq 2$, but $g_1(x)$ is not differentiable at $c = 0$.

Exercise 4.
a.) Prove:

Theorem. If $f\colon A \to \mathbb{R}$ is differentiable at $c \in A$, then f is continuous at $c \in A$.

b.) Show that the converse of this statement is false. In other words, show that f being continuous at $c \in A$ does not imply that f is differentiable at $c \in A$.

Exercise 5. Prove:

Theorem. Let $f\colon A \to \mathbb{R}$ and $g\colon A \to \mathbb{R}$, and assume that f and g are differentiable at $c \in A$. Then,
(i) $(f+g)'(c) = f'(c) + g'(c)$;
(ii) $(kf)'(c) = kf'(c)$ for all $k \in \mathbb{R}$;
(iii) $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$;
(iv) $(f/g)'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{g(c)^2}$ provided $g(c) \neq 0$.

Exercise 6. Prove:

Theorem.
Let $f\colon A \to \mathbb{R}$ and $g\colon B \to \mathbb{R}$ satisfy $f(A) \subset B$. If f is differentiable at $c \in A$ and g is differentiable at $f(c) \in B$, then $(g \circ f)'(c) = g'(f(c))f'(c)$.

Exercise 7.

a.) Prove:

Theorem (Interior extremum theorem)**.** Let f be differentiable function on (a, b). If f attains a maximum value at some point $c \in (a, b)$, then $f'(c) = 0$. Similarly, if f attains a minimal value at a point $c \in (a, b)$, then $f'(c) = 0$.

b.) Show that this theorem is not true for differentiable functions over closed intervals. That is, if f is differentiable on $[a, b]$, and f attains a maximum or minimum at $c \in [a, b]$, then $f'(c)$ is not necessarily equal to 0.

Exercise 8. Prove:

Theorem (Darboux's theorem)**.** If f is differentiable on $[a, b]$, and if $f'(a) < L < f'(b)$ or $f'(a) > L > f'(b)$, then there exists $c \in (a, b)$ such that $f'(c) = L$.