Analysis — Spring 2015
CU Boulder Math 3001

worksheet 14

Read section: 5.3

Exercise 1. Prove:

Theorem (Rolle's theorem)**.** Let $f\colon [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). If $f(a) = f(b)$, then there exists a point $c \in (a,b)$ where $f'(c) = 0$.

Exercise 2. Prove:

Theorem (Mean value theorem)**.** If $f\colon [a,b] \to \mathbb{R}$ is continuous and differentiable on (a,b), then there exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Exercise 3. a.) Suppose $g\colon A \to \mathbb{R}$ is differentiable on an interval A and satisfies $g'(x) = 0$ for all $x \in A$. Prove that $g(x) = k$ is a constant function.

b.) Suppose that $f(x)$ and $g(x)$ are differentiable functions on an interval A, and $f'(x) = g'(x)$ for all $x \in A$. Prove that $f(x) = g(x) + k$ for some constant $k \in \mathbb{R}$.

Exercise 4. Recall that $f\colon A \to \mathbb{R}$ is Lipschitz continuous on A if there exists $M > 0$ such that

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq M$$

for all $x, y \in A$. Show that if f is differentiable on a closed interval $[a,b]$ and if f' is continuous on $[a,b]$, then f is Lipschitz on $[a,b]$.

Exercise 5. Let h be a differentiable function defined on the interval $[0,3]$, and assume that $h(0) = 1$, $h(1) = 2$, and $h(3) = 2$.

a.) Prove that there exists a point $d \in [0,3]$ where $h(d) = d$.
b.) Prove that there exists a point $c \in [0,3]$ where $h'(c) = 1/3$.
c.) Prove that there exists a point $e \in [0,3]$ where $h'(e) = 1/4$.

Exercise 6. A *fixed point* of a function is a value x where $f(x) = x$. Prove that if f is differentiable on an interval and $f'(x) \neq 1$, then f has at most one fixed point.