Analysis — Spring 2015
CU Boulder Math 3001

worksheet 15

Read section: 7.2

Definition 1. A *partition* P of $[a, b]$ is a finite, ordered set

$$P = \{a = x_0 < x_1 < \cdots < x_n = b\}.$$

Definition 2. For each subinterval, let

$$m_k = \inf\{f(x) \colon x \in [x_{k-1}, x_k]\}$$
$$M_k = \sup\{f(x) \colon x \in [x_{k-1}, x_k]\}.$$

The *lower sum* of f with respect to P is

$$L(f, P) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}).$$

The *upper sum* of f with respect to P is

$$U(f, P) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}).$$

Definition 3. A partition Q is a *refinement* of P if Q contains all the points of P, i.e. $P \subseteq Q$.

Exercise 4. Prove that if $P \subseteq Q$, then $L(f, P) \leq L(f, Q)$, and $U(f, P) \geq U(f, Q)$.

Exercise 5. Prove that if P_1 and P_2 are any two partitions of $[a, b]$, then $L(f, P_1) \leq U(f, P_2)$.

Definition 6. Let $\mathcal{P}$ be the set of all partitions of $[a, b]$. The *lower integral* of f on $[a, b]$ is

$$L(f) = \sup\{L(f, P) \colon P \in \mathcal{P}\},$$

and the *upper integral* of f on $[a, b]$ is

$$U(f) = \inf\{U(f, P) \colon P \in \mathcal{P}\}.$$

Exercise 7. Prove that if f is a bounded function on $[a, b]$, then $L(f) \leq U(f)$.

Definition 8. A bounded function f on $[a, b]$ is *Riemann-integrable* if $U(f) = L(f)$, and we denote this value by $\int_a^b f$, i.e. $\int_a^b f = U(f) = L(f)$.

Exercise 9. Prove:

Theorem. A bounded function f is integrable on $[a, b]$ if and only if, for every $\epsilon > 0$, there exists a partition P_ϵ of $[a, b]$ such that $U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$.

Exercise 10. Prove:

Theorem. If f is continuous on $[a, b]$, then it is integrable.

Exercise 11. Let $f(x) = 2x + 1$ over the interval $[1, 3]$. Let $P = \{1, 3/2, 2, 3\}$.
a.) Compute $L(f, P)$, $U(f, P)$, and $U(f, P) - L(f, P)$.
b.) Find a partition P' of $[1, 3]$ for which $U(f, P') - L(f, P') < 2$.

Exercise 12. Prove that the constant function $f(x) = k$ is integral on $[a, b]$, and find the value $\int_a^b f$.