Analysis — Spring 2015
CU Boulder Math 3001

worksheet 16

Read sections: 7.3–7.5

Exercise 1.

a.) Let $f_1 \colon [0,1] \to \mathbb{R}$ be the function defined by

$$f_1(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x \neq 1. \end{cases}$$

Prove that f_1 is integrable on $[0,1]$ and compute $\int_0^1 f_1$.

b.) Let A be a finite set of values in $[0,1]$, and define $f_2(x)$ by

$$f_2(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Prove that f_2 is integrable on $[0,1]$ and compute $\int_0^1 f_2$.

c.) Let $f_3 \colon [0,1] \to \mathbb{R}$ be the function defined by

$$f_3(x) = \begin{cases} 1 & \text{if } x = 1/n \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Prove that f_3 is integrable on $[0,1]$ and compute $\int_0^1 f_3$.

d.) Let B be an infinite set of values in $[0,1]$, and define $f_4(x)$ by

$$f_4(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B. \end{cases}$$

Prove that f_4 may not be integrable on $[0,1]$.

Exercise 2. Prove:

Theorem. Suppose $f \colon [a,b] \to \mathbb{R}$ is bounded, and let $c \in (a,b)$. Then f is integrable on $[a,b]$ if and only if f is integrable on $[a,c]$ and $[c,b]$, i.e. $\int_a^b f = \int_a^c f + \int_c^b f$.

Exercise 3. Suppose that f is a bounded function on $[a,b]$. Prove that f is integrable on $[a,b]$ if and only if there exists a sequence of partitions (P_n) satisfying

$$\lim_{n\to\infty} [U(f,P_n) - L(f,P_n)] = 0.$$

Exercise 4. Prove:

Theorem. Assume f and g are integrable functions on the interval $[a,b]$.

a.) The function $f+g$ is integrable on $[a,b]$ with $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

b.) For $k \in \mathbb{R}$, the function kf is integrable with $\int_a^b kf = k\int_a^b f$.

c.) If $m \leq f \leq M$, then $m(b-a) \leq \int_a^b f \leq M(b-a)$.

d.) If $f \leq g$, then $\int_a^b f \leq \int_a^b g$.

e.) The function $|f|$ is integrable and $|\int_a^b f| \leq \int_a^b |f|$.

Exercise 5. Prove:

Theorem (Fundamental theorem of calculus.)**.**
a.) If $f\colon [a,b] \to \mathbb{R}$ is integrable, and $F\colon [a,b] \to \mathbb{R}$ satisfies $F'(x) = f(x)$ for all $x \in [a,b]$, then

$$\int_a^b f = F(b) - F(a).$$

b.) Let $f\colon [a,b] \to \mathbb{R}$ be integrable, and define

$$G(x) = \int_a^x g$$

for all $x \in [a,b]$. Then G is continuous on $[a,b]$. If g is continuous at $c \in [a,b]$, then G is differentiable at c and $G'(c) = g(c)$.

Exercise 6. Let $f(x) = |x|$, and define $F(x) = \int_{-1}^x f$. Find a formula for $F(x)$ for all x. Where is F continuous? Where is F differentiable? Where does $F'(x) = f(x)$?

Exercise 7. For $x > 0$, let

$$H(x) = \int_1^x \frac{1}{t}\, dt.$$

a.) What is $H(1)$? Find $H'(x)$.
b.) Show that H is *strictly increasing*, i.e. show that if $0 < x < y$, then $H(x) < H(y)$.
c.) Show that $H(cx) = H(c) + H(x)$.

Exercise 8. If g is continuous on $[a,b]$, show that there exists a point $c \in (a,b)$ where

$$g(c) = \frac{1}{b-a} \int_a^b g.$$