Analysis — Spring 2015
CU Boulder Math 3001

worksheet 17

Read section: 6.2

Definition 1. Suppose $f_1, f_2, f_3, \ldots$ is a set of functions that share a common domain A, i.e. $f_n\colon A \to \mathbb{R}$ for each $n \in \mathbb{N}$. The sequence of functions (f_n) *converges pointwise on* A to a function $f\colon A \to \mathbb{R}$ if the sequence $(f_n(a))$ converges to $f(a)$ for all $a \in A$.

Exercise 2. Let $f_n(x) = x^2/n$. Prove that (f_n) converges pointwise to a function f on $\mathbb{R}$. What is f?

Exercise 3. Let $f_n(x) = x^n$. Prove that (f_n) converges pointwise to a function f on $[-1, 1]$. What is f? What happens to (f_n) outside the interval $[-1, 1]$?

Exercise 4. Let $f_n(x) = x^{1+1/(2n-1)}$. Prove that (f_n) converges pointwise to a function f on the set $[-1, 1]$. What is f?

Definition 5. Let (f_n) be a sequence of function with a common domain A. Then (f_n) *converges uniformly on* A to a function f if, for each $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon$ whenever $n \geq N$ and $x \in A$.

Exercise 6. Let $f_n = \frac{1}{n(x^2+1)}$. Prove that (f_n) converges uniformly on $\mathbb{R}$ to a function f. What is f?

Exercise 7. Let $f_n(x) = \frac{1}{(x-n)^2+1}$. Does (f_n) converge pointwise on $\mathbb{R}$? Does (f_n) converge uniformly on $\mathbb{R}$?

Exercise 8. Prove:

Theorem (Cauchy criterion for uniform convergence)**.** A sequence of functions (f_n) converges uniformly on A if and only if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| < \epsilon$ whenever $n, m \geq N$ and $x \in A$.

Exercise 9. Suppose (f_n) is a sequence of functions that converges uniformly to f on A. Prove that if each f_n is continuous at $c \in A$, then f is continuous at c.

Exercise 10. Suppose (f_n) is a sequence of uniformly continuous functions that converges uniformly to f on A. Prove that f is uniformly continuous on A.

Exercise 11. Suppose (f_n) is a sequence of bounded functions that converges uniformly to f on A. Prove that f is bounded.

Exercise 12. Suppose (f_n) and (g_n) are uniformly convergent sequences of functions.

a.) Prove that $(f_n + g_n)$ is uniformly convergent.

b.) Suppose there exists an $M > 0$ such that $|f_n| \leq M$ and $|g_n| \leq M$ for all $n \in \mathbb{N}$. Prove that $(f_n g_n)$ converges uniformly.

c.) Prove that, in general (i.e. when the bounded condition in (b) is removed), $(f_n g_n)$ does not converge uniformly.