Analysis — Spring 2015
CU Boulder Math 3001

worksheet 18

Read sections: 6.3 and 6.4

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**Exercise 1.** Prove:

**Theorem.** Let $(f_n)$ be a sequence of functions that converges pointwise to $f$ on the closed interval $[a,b]$, and assume that each $f_n$ is differentiable. If $(f_n')$ converges uniformly on $[a,b]$ to a function $g$, then the function $f$ is differentiable and $f' = g$

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**Exercise 2.** Let $h_n(x) = \sin(nx)/n$.
a.) Show that $(h_n) \to 0$ uniformly on $\mathbb{R}$.
b.) At what points does the sequence of $(h'(x))$ converge?

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**Exercise 3.** Let $g_n(x) = x^n/n$.
a.) Show that $(g_n)$ converges uniformly on $[0,1]$ and find $g = \lim g_n$.
b.) Show that $g$ is differentiable and compute $g'(x)$ for all $x \in [0,1]$.
c.) Show that $(g_n')$ converges uniformly on $[0,1]$. Is the convergence uniform?
d.) Let $h = \lim g_n'$. Does $h = g'$?

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**Definition 4.** For each $n \in \mathbb{N}$, let $f_n$ and $f$ be functions defined on a set $A \subseteq \mathbb{R}$. Consider the infinite series

$$\sum_{n=1}^{\infty} f_n(x) = f_1(x) + f_2(x) + f_3(x) + \cdots , \tag{1}$$

and define

$$s_k(x) = f_1(x) + f_2(x) + \cdots + f_k(x) \tag{2}$$

to be the $k$-th partial sum.

The infinite series (1) *converges pointwise* on $A$ to $f(x)$ if the sequence of partial sums $(s_k(x))$ converges pointwise to $f$ on $A$.

The infinite series *converges uniformly* on $A$ to $f(x)$ if the sequence of partial sums $(s_k(x))$ converges uniformly to $f$ on $A$.

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**Exercise 5.** Prove:

**Theorem.** Let $(f_n)$ be a sequence of continuous functions defined on a set $A \subseteq \mathbb{R}$, and assume $\sum_{n=1}^{\infty} f_n$ converges uniformly to a function $f$ on $A$. Then $f$ is continuous on $A$.

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**Exercise 6.** Prove:

**Theorem.** A series $\sum_{n=1}^{\infty} f_n$ converges uniformly on $A \subseteq \mathbb{R}$ if and only if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n > m \geq N$,

$$|f_{m+1}(x) + f_{m+2}(x) + \cdots + f_n(x)| < \epsilon$$

for all $x \in A$.

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**Exercise 7.** Prove:

**Theorem.** For each $n \in \mathbb{N}$, let $f_n$ be a function defined on a set $A \subseteq \mathbb{R}$, and let $M_n > 0$ be a real number satisfying $|f_n(x)| \leq M_n$ for all $x \in A$. If $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n$ converges uniformly on $A$.

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**Exercise 8.** Prove that if $\sum_{n=1}^{\infty} f_n$ converges uniformly, then $(f_n)$ converges uniformly to 0.

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**Exercise 9.** Prove that $g(x) = \sum_{n=1}^{\infty} \cos(2^n x)/2^n$ is continuous on all of $\mathbb{R}$.

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**Exercise 10.** Prove that $h(x) = \sum_{n=1}^{\infty} x^n/n^2$ is continuous on $[-1, 1]$.