Analysis — Spring 2015
CU Boulder Math 3001

worksheet 4 supplement

1. logical statements

When trying to understand logical statements (such as the definition of convergence), it often helps to think of the variables in these statements to real objects. Consider two sets $\{c_i\}$ and $\{s_i\}$, where $\{c_i\}$ is a set of circles of various colors, and $\{s_i\}$ is a set of squares of various colors. For example,

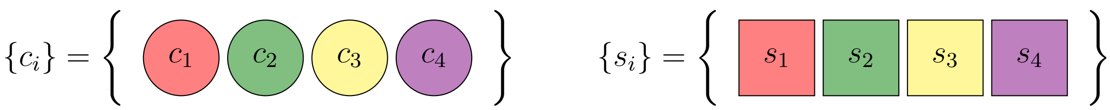


Figure 1

When we write $|c_i - s_j| < \epsilon$, we will take that to mean that the circle c_i and the square s_j are the same color.

For example, the statement

$$|c_i - s_j| < \epsilon \text{ if and only if } i = j \tag{1}$$

is representative of Figure 1: a circle and square are the same color if and only if they are labeled with the same number.

Let's look at another statement:

$$\text{For each } i \text{ there exists a } j \text{ such that } |c_i - s_j| < \epsilon. \tag{2}$$

Translation: "for each circle there is a square of the same color." Or to put it another way: "each circle shares a color with at least one of the squares."

Figure 1 also satisfies statement (2), but statement (1) is a much stronger condition than statement (2). Figures 2 and Figure 3 are examples of sets that satisfy statement (2), but not statement (1).

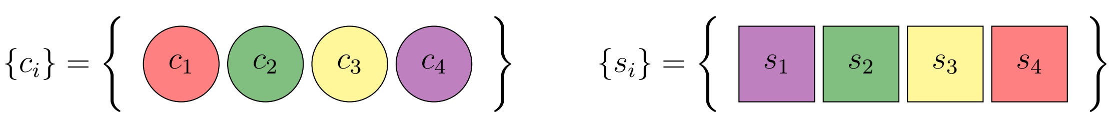


Figure 2

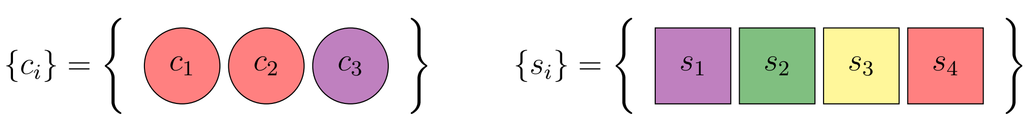
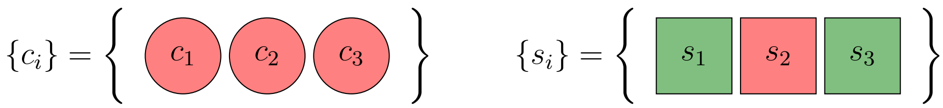


Figure 3

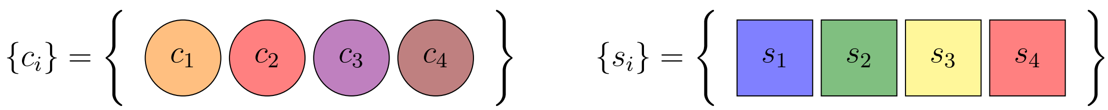
Exercise 1. Match the pictures to the statements. There may be more than one picture for each statement. Some statements may not be satisfied by any of the pictures.

(I.) For each i and each j, $|c_i - s_j| < \epsilon$.
(II.) For each j there exists an i such that $|c_i - s_j| < \epsilon$.
(III.) There exists an i and there exists a j such that $|c_i - s_j| < \epsilon$.
(IV.) There exists a j such that $|c_i - s_j| < \epsilon$ for all i.

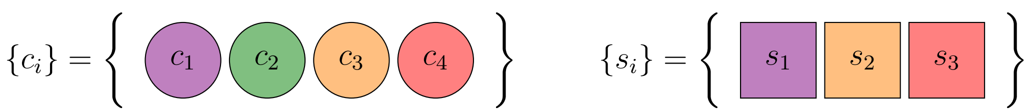
(A)



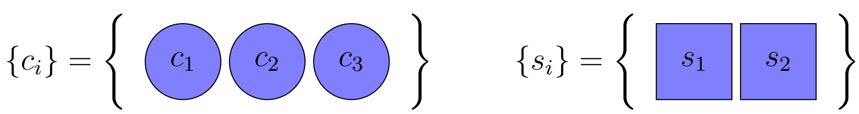
(B)



(C)



(D)

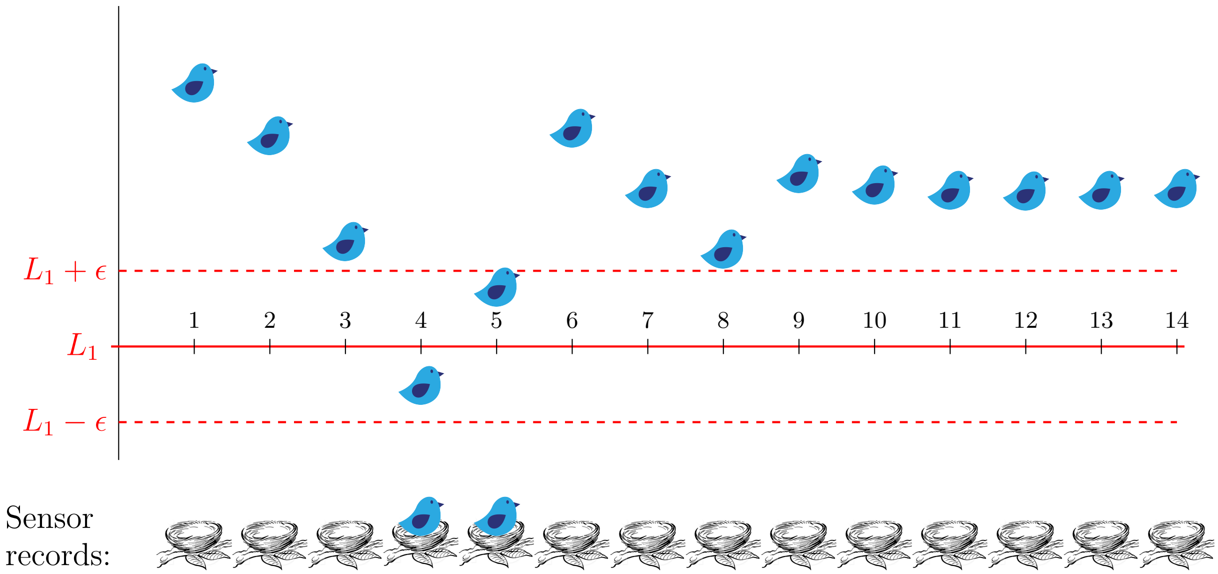


2. Definition of Convergence

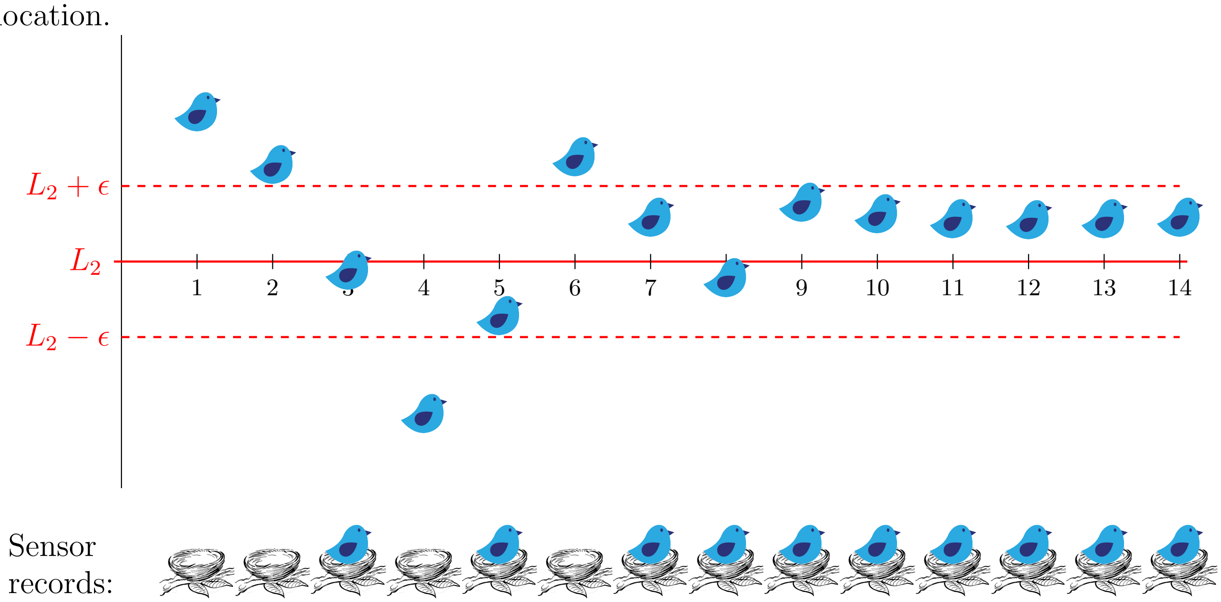
In order to understand the statement of convergence, we have to change our story slightly. Suppose we are out on a hike and we see a bird flying about in a forest full of birds' nests. We want to decide to figure out which nest belongs to this particular bird, so we pick a nest and set up a sensor will tell us if the bird is in that particular nest, or away from that nest. The sensor is very simple: if the bird comes within the sensor's range, then the sensor will record that the bird is in the nest (whether or not the bird is actually in the nest). Otherwise, the bird is out of the nest. To make this precise, let

- L be the position of the nest we have chosen,
- a_n be the position of the bird at time n, and
- ϵ be the range of the sensor.

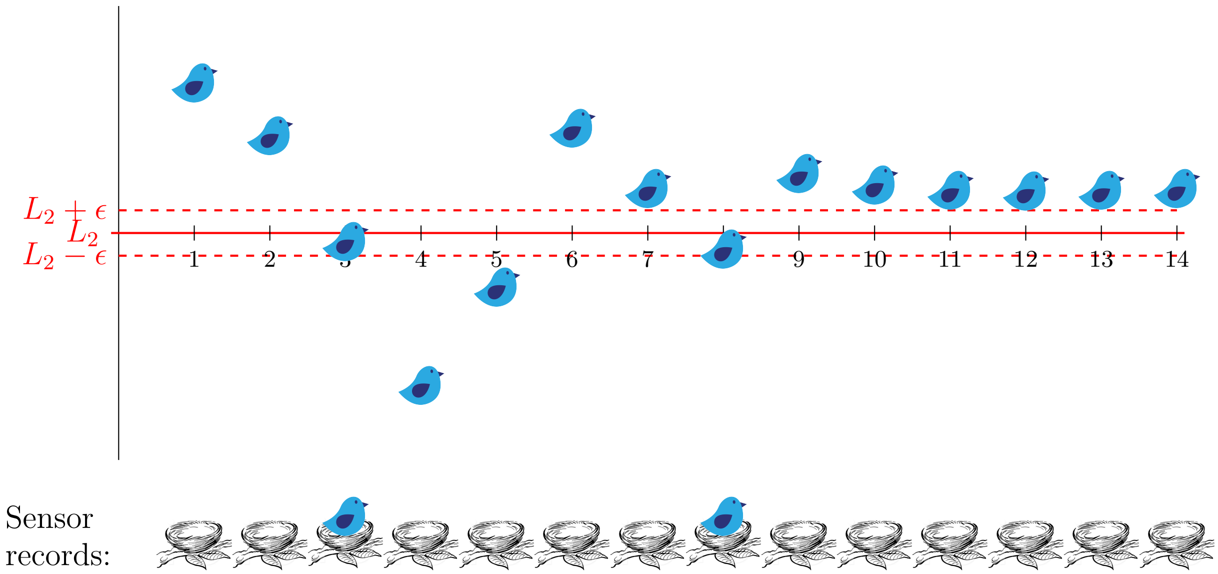
The sensor records that the bird is in the nest at time n if $|a_n - L| < \epsilon$ (the bird is near the nest). We decide that if the bird is always in the nest after a certain time N, then the nest must belong to the bird. That is, for L to be the position of the nest, there must be an N such that $|a_n - L| < \epsilon$ whenever $n \geq N$.



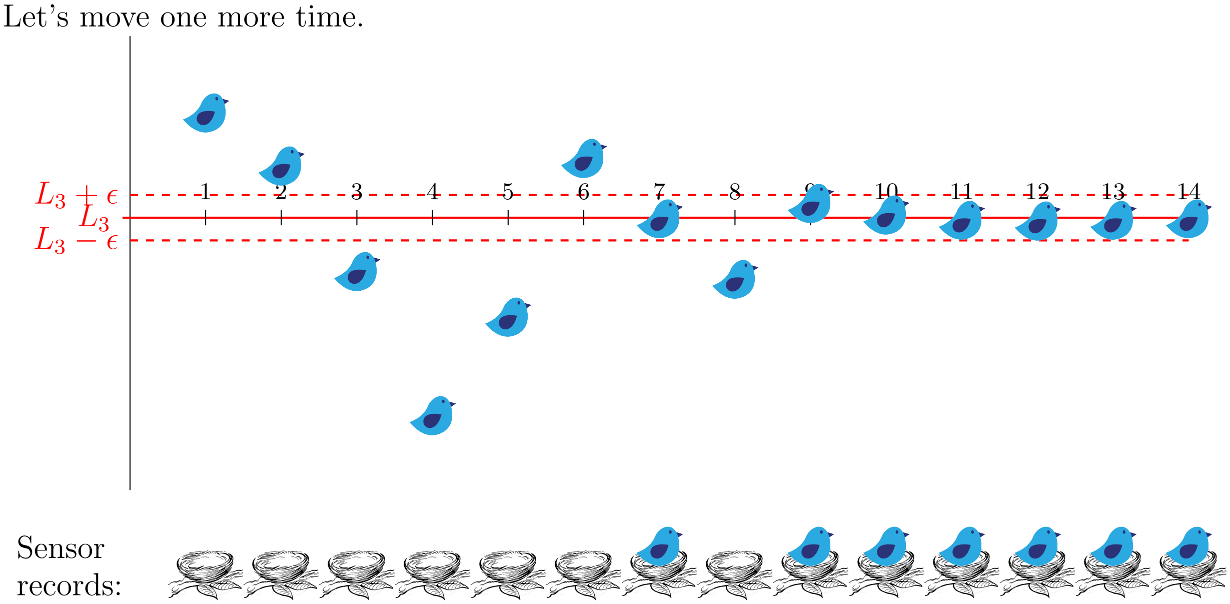
Our first guess of the location of the bird's nest (L_1) appears to be incorrect. The sensor barely records that the bird is in the nest. Mathematically speaking, there does not (appear to) exist an N such that N such that $|a_n - L_1| < \epsilon$ whenever $n \geq N$. Let's try a different location.



This is more promising, the bird is always recorded to be in the nest after time $N = 7$. That is, $|a_n - L_2| < \epsilon$ whenever $n \geq 7$. However, since the sensor does not know that the bird is in the nest, it might be that the bird's true nest is simply near enough to trigger the sensor. To test this hypothesis, we should decrease the sensitivity of the sensor.



Now that we've decreased the sensitivity, we see that L_2 cannot be the location of the nest. Let's move one more time.



Now we are back on the right track. The bird is always in the nest after time $N = 9$. But once again, we cannot be sure that L_3 is the location of the bird's nest unless we continue the reduce the sensor's sensitivity and still have the sensor record the bird in the nest. In other words, the location of the nest is L if for each possible sensitivity setting ϵ, there is a time N after which (whenever the time n is $\geq N$) the bird is recorded to be in the nest ($|a_n - L| < \epsilon$).

Let's revisit problem 7 on worksheet 4. Statement (a) says: *A sequence* (a_n) *converges to* $a \in \mathbb{R}$ *if there exists* $\epsilon > 0$ *and there exists* $n \in \mathbb{N}$ *such that* $|a_n - a| < \epsilon$. Putting this into the language of our story, the statement says: "the location of the nest is a if there exists a sensitivity setting ϵ and a time n such that the bird is in the nest at time n." According to this statement, L_1, L_2, and L_3 are all locations for the bird's nest. (Why?)