Analysis — Spring 2015
CU Boulder Math 3001

worksheet 8

Read section: 3.2

Definition. Given $a \in \mathbb{R}$ and $\epsilon > 0$, the *ϵ-neighborhood of a* is the open interval

$$V_\epsilon(a) = \{x \in \mathbb{R} \colon |x - a| < \epsilon\}.$$

Definition. A set $A \subseteq \mathbb{R}$ is *open* if for all points $a \in A$, there exists an ϵ such that $V_\epsilon(a) \subseteq A$.

Exercise 1. Show that each of the following sets is open.

(a) $\mathbb{R}$ (b) (a, b) (c) $(a, b) \cup (c, d)$ (d) $\emptyset$

Exercise 2.
(a) Prove that the union of arbitrarily many open sets is open.
(b) Prove that the intersection of finitely many open sets is open.

Exercise 3. Given an example of an infinite collection of nested open sets whose intersection is closed and nonempty.

Definition. A point a is a *limit point* of a set A if every ϵ-neighborhood $V_\epsilon(a)$ intersects A in at least one point other than a.

Exercise 4. Find the limit points of each of the following sets.

(a) $\mathbb{R}$
(b) $\mathbb{Q}$
(c) $\mathbb{Z}$
(d) $\left\{\frac{1}{n} \colon n \in \mathbb{N}\right\}$
(e) $(0, 1) \cap \{2\}$
(f) (a, b)
(g) $[a, b]$
(h) $\sum_{n=1}^{\infty} 3^{-n}$

Exercise 5. Prove that a point a is a limit point of a set A if and only if $a = \lim a_n$ for some sequence (a_n) contained in A satisfying $a_n \neq a$ for all $n \in \mathbb{N}$.

Definition. A point $a \in A$ is a *isolated point* of A if it is not a limit point of A.

Exercise 6. Identify all the isolated points of the following sets.

(a) $\emptyset$
(b) $\mathbb{Z}$
(c) $\mathbb{Q}$
(d) $\mathbb{R}$
(e) (a, b)
(f) $\{n^{-1} \colon n \in \mathbb{N}\}$

Exercise 7. Prove that a is an isolated point of A if and only if there exists an open neighborhood $V_\epsilon(a)$ satisfying $V_\epsilon(a) \cap A = \{a\}$.

Definition. A set $A \subseteq \mathbb{R}$ is *closed* if it contains its limit points.

Exercise 8. Which of the following sets are closed?

(a) $\emptyset$	(c) $\mathbb{Q}$	(e) $[a, b]$	(g) $(a, b]$
(b) $\mathbb{Z}$	(d) $\mathbb{R}$	(f) (a, b)	(h) $\{n^{-1} : n \in \mathbb{N}\}$

Definition. Let $A \subseteq \mathbb{R}$, and let L be the set of limit points of A. The *closure* of A is the set $\overline{A} = A \cup L$.

Exercise 9. Determine the closure of each of the following sets.

(a) $\emptyset$	(c) $\mathbb{Q}$	(e) $[a, b]$	(g) $(a, b]$
(b) $\mathbb{Z}$	(d) $\mathbb{R}$	(f) (a, b)	(h) $\{n^{-1} : n \in \mathbb{N}\}$

Exercise 10. Given $A \subseteq \mathbb{R}$, let L be the set of limit points of A.

(a) Prove that L is closed.
(b) Prove that if a is a limit point of $\overline{A}$, then a is a limit point of A.
(c) Prove that $\overline{A}$ is closed.
(d) Prove that $\overline{A}$ is the smallest closed set containing A. That is, prove that if U is a closed set containing A, then $\overline{A} \subseteq U$.

Definition. Recall that the *complement* of a set A is the set

$$A^c = \{x \in \mathbb{R} : x \notin A\}.$$

Prove that a set A is open if and only if A^c is closed.

Exercise 11.

(a) Prove that the union of finitely many closed sets is closed.
(b) Prove that the intersection of arbitrarily many closed sets is closed.

Exercise 12.

(a) Suppose that a is a limit point of $A \cup B$. Prove that a is a limit point of A or B (or both).
(b) Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.