Analysis — Spring 2015
CU Boulder Math 3001

worksheet 9

Read section: 3.3

Definition. A set $A \subseteq \mathbb{R}$ is *compact* if every sequence in K has a subsequence that converges to a limit that is also in K.

Exercise 1. Use the definition of compact to determine whether or not each of the following sets is compact.

(a) $\mathbb{R}$
(b) $\mathbb{Q} \cap [a, b]$
(c) $\mathbb{Z} \cap [a, b]$
(d) $[a, b]$
(e) (a, b)
(f) $\{n^{-1} : n \in \mathbb{N}\}$

Exercise 2. Prove that if A is a closed and bounded set, then $\sup A$ and $\inf A$ exist and are contained in A.

Exercise 3. Prove the Heine–Borel Theorem.

Theorem (Heine–Borel). A set $A \subseteq \mathbb{R}$ is compact if and only if it is closed and bounded.

Definition. Let $A \subseteq \mathbb{R}$. An *open cover* for A is a collection of open sets $U = \{U_\lambda : \lambda \in \Lambda\}$ whose union contains the set A. A *finite subcover* is a finite subcollection of open sets in U whose union contains A.

Exercise 4. Prove that $U = \{(n, n+2) : n \in \mathbb{N}\}$ is an open cover of $\mathbb{R}$. Prove that U does not contain a finite subcover of $\mathbb{R}$.

Exercise 5. Prove that $U = \{(0, x) : x \in (0, 1)\}$ is an open cover for $A = (0, 1)$. Prove that U does not contain a finite subcover of A.

Exercise 6. Prove that if $A \subseteq \mathbb{R}$ is a compact set, then A is compact if and only if every open cover of A contains a finite subcover of A.

Exercise 7. Suppose that $A = \{A_\lambda : \lambda \in \Lambda\}$ is a collection of compact sets, with the property that any intersection of finitely many of these sets is nonempty. Prove that $\bigcap_{\lambda \in \Lambda} A_\lambda$ is nonempty.

Exercise 8. Let A be the union of finitely many compact sets. Prove that A is compact by showing that every open cover of A contains a finite subcover.

Exercise 9. Prove that if $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$ is a nested sequence of compact sets, then $\bigcap_{n=1}^{\infty} A_n$ is nonempty.

Exercise 10. Let A be the intersection of arbitrarily many compact sets. Prove that A is compact by showing that every open cover of A contains a finite subcover.

Exercise 11. Prove that if A is compact and B is closed, then $A \cap B$ is compact.