12.4: THE CROSS PRODUCT

The cross product, $\vec{a} \times \vec{b}$, is another way to multiply vectors. Unlike the dot product, the cross product is a <u>vector</u> and is only defined for three-dimensional vectors.

Definition 1. If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, then the **cross product** of $\vec{a}$ and $\vec{b}$ is the vector

$$\vec{a} \times \vec{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle.$$

We can also define the cross product in terms of determinants. A **determinant of order 2** is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

For instance, $\begin{vmatrix} 7 & 2 \\ 3 & 1 \end{vmatrix} =$

A **determinant of order 3** can be defined in terms of second-order determinants:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

For example, $\begin{vmatrix} 1 & -2 & 4 \\ 3 & 0 & 2 \\ 5 & 1 & -2 \end{vmatrix} =$

We can then write the cross product of $\vec{a}$ and $\vec{b}$ as

$$\vec{a} \times \vec{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}, \qquad \text{or}$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

<u>**Determinant shortcut:**</u> (diagonals method)

Example 1. Find the cross product of $\vec{a} = \langle 1, 2, -3 \rangle$ and $\vec{b} = \langle 2, -3, -5 \rangle$.

Solution.

Theorem 1. *The vector* $\vec{a} \times \vec{b}$ *is orthogonal to both* $\vec{a}$ *and* $\vec{b}$*.*

Proof. Exercise. You need to show both $(\vec{a} \times \vec{b}) \cdot \vec{a} = \vec{0}$ and $(\vec{a} \times \vec{b}) \cdot \vec{b} = \vec{0}$.

$(\vec{a} \times \vec{b}) \cdot \vec{a} =$

$(\vec{a} \times \vec{b}) \cdot \vec{b} =$

□

Right-hand rule:

Theorem 2. *If θ is the angle between $\vec{a}$ and $\vec{b}$, with $0 \leq \theta \leq \pi$, then*

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin \theta.$$

Proof. See the text. □

Corollary 3. *Two non-zero vectors $\vec{a}$ and $\vec{b}$ are parallel if and only if $\vec{a} \times \vec{b} = \vec{0}$. In particular, $\vec{a} \times \vec{a} = \vec{0}$.*

Proof. The vectors $\vec{a}$ and $\vec{b}$ are parallel if and only if the angle between them is $\theta = 0$ or π. In either case $\sin \theta = 0$, so $|\vec{a} \times \vec{b}| = 0$ and thus $\vec{a} \times \vec{b} = \vec{0}$. □

Fact: The length of the cross product, $|\vec{a} \times \vec{b}|$, is equal to the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$.

Example 2. Find a vector perpendicular to the plane that passes through the points $P = (1, -2, 5)$, $Q = (3, 7, 1)$, and $R = (-2, -1, 1)$. Then find the area of the triangle $\triangle PQR$.

Solution.

The cross products of the standard basis vectors:

Properties: If $\vec{u}, \vec{v}$, and $\vec{w}$ are vectors in V_3 and c is a scalar, then

(1) $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$

(2) $(c\vec{u}) \times \vec{v} = c(\vec{u} \times \vec{v}) = \vec{u} \times (c\vec{v})$

(3) $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$

(4) $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$

(5) $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$

(6) $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$

Definition 2. The **scalar triple product** of the vectors $\vec{a}, \vec{b}$, and $\vec{c}$ is

$$\begin{aligned} \vec{a} \cdot (\vec{b} \times \vec{c}) &= a \cdot \left(\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} \vec{i} - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} \vec{j} + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \vec{k} \right) \\ &= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \end{aligned}$$

Geometric significance:

Theorem 4. *The volume of the parallelepiped determined by $\vec{a}, \vec{b}$, and $\vec{c}$ is the magnitude of their scalar triple product:*

$$V = |\vec{a} \cdot (\vec{b} \times \vec{c})|.$$

<u>Fact:</u> Three vectors $\vec{a}, \vec{b}$, and $\vec{c}$ are **coplanar** (they lie in the same plane) if

$$|\vec{a} \cdot (\vec{b} \times \vec{c})| = 0.$$

Why?

Example 3. Show that $\vec{a} = \langle 1, 4, -7 \rangle$, $\vec{b} = \langle 2, -1, 4 \rangle$, and $\vec{c} = \langle 0, -9, 18 \rangle$ are coplanar.

<u>Solution.</u>