12.5: EQUATIONS OF LINES AND PLANES

A line is determined by a point and a direction. In $\mathbb{R}^2$ this amounts to a point and a slope.

Let $P_0(x_0, y_0, z_0)$ be a point on a line L in $\mathbb{R}^3$. Suppose the direction of L is parallel to a vector $\vec{v} = \langle a, b, c \rangle$. We can derive an equation for the line L.

Example 1.

(a) Find a vector equation and parametric equations for the line L through the point $(5, -3, 4)$ and parallel to the vector $2\vec{i} - 5\vec{j} - \vec{k}$.

(b) Find two other points on the line.

Solution.

__**Note.**__ Vector and parametric equations for a line are not unique. Why?

Definition 1. Given a line L with parametric equations

$$\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned}$$

then, if a, b, and c are all non-zero, we can solve for t to obtain the **symmetric equations** of L:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

If one or more of a, b, or c are zero we can still obtain symmetric equations for L. For instance, if $a = 0$ then

$$x = x_0, \quad \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

This means that L lies in the vertical plane $x = x_0$.

Example 2.

(a) Find parametric and symmetric equations of the line through the points $A(-1, 3, 6)$ and $B(2, 1, 1)$.

(b) At what point does this line intersect the xy-plane?

Solution.

Note. The symmetric equations for a line through the points (x_0, y_0, z_0) and (x_1, y_1, z_1) are

We sometimes want to describe a line segment rather than an entire line. We can do so by restricting the values of t. For instance, in Example 2,

Definition 2. Two lines are **skew** if they do not intersect and are not parallel. Skew lines can not lie in the same plane.

Example 3. Show that the lines L_1 and L_2 with parametric equations

$$\begin{aligned} x &= 2 + t \qquad & y &= -3 + 2t \qquad & z &= 1 - t \\ x &= 3 - 2s \qquad & y &= -1 - 2s \qquad & z &= 4s + 5 \end{aligned}$$

are skew.

Solution.

Planes:

We need two things to completely describe a plane:

- a point $P_0(x_0, y_0, z_0)$ in the plane,
- the direction orthogonal to the plane, given by a vector $\vec{n} = \langle a, b, c \rangle$, called the **normal vector**.

We seek and equation which must be satisfied by any point $P(x, y, z)$ on the plane.

Example 4. Find an equation of the plane through the point $(-3, 4, 1)$ which has normal vector $\vec{n} = \langle 1, -1, 2 \rangle$.

Solution.

Example 5. Find an equation of the plane which passes through the points $P(2, 1, 4)$, $Q(-1, 4, 6)$, and $R(5, 2, 0)$.

Solution.

Example 6. Find the point where the line with parametric equations

$$x = -3 - t \qquad y = 1 - t \qquad z = 4 + 2t$$

intersects the plane $2x + 5y - 3z = 13$.

Solution.

Definition 3. Two planes are parallel if their normal vectors are parallel.

Example 7. The planes

$$-x + 4y + 3z = 2$$

and

$$2x + 8y - 6z = 10$$

are parallel, because their normal vectors, $\langle -1, 4, 3\rangle$ and $\langle 2, 8, -6\rangle$, are parallel.

Definition 4. If two planes are not parallel then they intersect in a straight line, and the **angle between the planes** is defined to be the acute angle θ between their normal vectors. If $\theta = \pi/2$ radians then the planes are **orthogonal**.

Example 8.

(a) Find the angle between the planes $2x + 3y - z = 1$ and $x - 2y + z = 6$.

(b) Find symmetric equations for the line of intersection of the above planes.

Solution.

Definition 5. The distance D from the point $P_1(x_1, y_1, z_1)$ to the plane $\mathcal{P}$ with equation $ax+by+cz+d=0$ is the length of the shortest line segment connecting P_1 to the plane. We can find an explicit formula for D:

Definition 6. The distance between two parallel planes is the distance from one plane to any point on the other plane.

Example 9. Find the distance between the planes $-3x+7y+z=2$ and $9x-21y-3z=10$.

Solution.