13.3 – Arc length and curvature University of Massachusetts Amherst Math 233 – Fall 2013

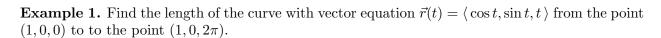
Recall. In Calc II, we saw that the arc length of a plane curve with parametric equations x = f(t), y = g(t), $a \le t \le b$ is given by

$$L = \int_{a}^{b} \sqrt{[f'(t)]^{2} + [g'(t)]^{2}} dt = \int_{a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^{2} + \left(\frac{dy}{dt}\right)^{2}} dt$$

We can extend this equation to curves in 3-space. If a curve has vector equation $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, $a \leq t \leq b$, or equivalently has parametric equations x = f(t), y = g(t), and z = h(t), with $a \leq t \leq b$, then if f', g', and h' are continuous, the arc length of the curve is

$$L = \int_{a}^{b} \sqrt{[f'(t)]^{2} + [g'(t)]^{2} + [h'(t)]^{2}} dt$$
$$= \int_{a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^{2} + \left(\frac{dy}{dt}\right)^{2} + \left(\frac{dz}{dt}\right)^{2}} dt$$
$$= \int_{a}^{b} |\vec{r}'(t)| dt$$

Interpretation of the formula.



Solution.

Note. A curve can be represented by more than one vector function, or parameterization. For instance,

Definition 1. The arc length function s(t) is given by

$$s(t) = \int_{a}^{t} |\vec{r}'(u)| \, du$$

By the Fundamental Theorem of Calculus,

$$\frac{ds}{dt} = |\vec{r}'(t)|$$

Curvature.

Definition 2. A parameterization $\vec{r}(t)$ is *smooth* on an interval if $\vec{r}'(t)$ is continuous and $\vec{r}'(t) \neq \vec{0}$ on that interval. A curve is *smooth* if it has a smooth parameterization.

Definition 3. If $\vec{r}(t)$ is a smooth parameterization of a curve, then its *unit tangent vector* is given by

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

Definition 4. The *curvature* of a curve $\vec{r}(t)$ is

$$\kappa(t) = \left| \frac{d\vec{T}}{ds} \right| = \frac{|d\vec{T}/dt|}{|ds/dt|} = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$$

Example 2. Find the curvature of a circle of radius a.

Solution.

Note. From 13.2, since $|\vec{T}(t)| = 1$ we know that $\vec{T}(t) \cdot \vec{T}'(t) = 0$, so $\vec{T}'(t)$ is orthogonal (or normal) to $\vec{T}(t)$.

Definition 5. The *unit normal vector* to a curve $\vec{r}(t)$ is

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$$

The vector $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$ is called the *binormal vector*; it is normal to both T and N and is a unit vector. The plane determined by \vec{N} and \vec{B} at a point P on a curve C is called the *normal plane* of C at P.

Example 3. Find the unit normal and binormal vectors of $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$.

Solution.