13.3 – Arc length and curvature
University of Massachusetts Amherst
Math 233 – Fall 2013

Recall. In Calc II, we saw that the arc length of a plane curve with parametric equations $x = f(t)$, $y = g(t)$, $a \le t \le b$ is given by

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

We can extend this equation to curves in 3-space. If a curve has vector equation $\vec{r}(t) = \langle\, f(t), g(t), h(t)\,\rangle$, $a \le t \le b$, or equivalently has parametric equations $x = f(t)$, $y = g(t)$, and $z = h(t)$, with $a \le t \le b$, then if f', g', and h' are continuous, the *arc length* of the curve is

$$\begin{aligned} L &= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt \\ &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \\ &= \int_a^b |\vec{r}\,'(t)|\, dt \end{aligned}$$

Interpretation of the formula.

Example 1. Find the length of the curve with vector equation $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ from the point $(1, 0, 0)$ to to the point $(1, 0, 2\pi)$.

Solution.

Note. A curve can be represented by more than one vector function, or *parameterization*. For instance,

Definition 1. The *arc length function* $s(t)$ is given by

$$s(t) = \int_a^t |\vec{r}\,'(u)|\, du$$

By the Fundamental Theorem of Calculus,

$$\frac{ds}{dt} = |\vec{r}\,'(t)|$$

Curvature.

Definition 2. A parameterization $\vec{r}(t)$ is *smooth* on an interval if $\vec{r}\,'(t)$ is continuous and $\vec{r}\,'(t) \neq \vec{0}$ on that interval. A curve is *smooth* if it has a smooth parameterization.

Definition 3. If $\vec{r}(t)$ is a smooth parameterization of a curve, then its *unit tangent vector* is given by

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$

Definition 4. The *curvature* of a curve $\vec{r}(t)$ is

$$\kappa(t) = \left|\frac{d\vec{T}}{ds}\right| = \frac{|d\vec{T}/dt|}{|ds/dt|} = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|}$$

Example 2. Find the curvature of a circle of radius a.

Solution.

Note. From 13.2, since $|\vec{T}(t)| = 1$ we know that $\vec{T}(t) \cdot \vec{T}\,'(t) = 0$, so $\vec{T}\,'(t)$ is orthogonal (or normal) to $\vec{T}(t)$.

Definition 5. The *unit normal vector* to a curve $\vec{r}(t)$ is

$$\vec{N}(t) = \frac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|}$$

The vector $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$ is called the *binormal vector*; it is normal to both T and N and is a unit vector. The plane determined by $\vec{N}$ and $\vec{B}$ at a point P on a curve C is called the *normal plane* of C at P.

Example 3. Find the unit normal and binormal vectors of $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$.

Solution.