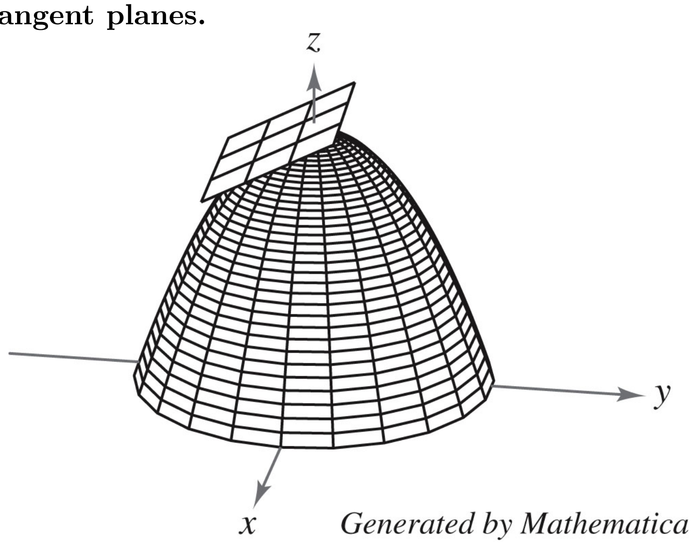
14.4 – Tangent planes and linear approximations
University of Massachusetts Amherst
Math 233 – Fall 2013

Tangent planes.



Definition 1. Let S be a surface with equation $z = f(x, y)$ where f has continuous first partial derivatives and suppose $P(x_0, y_0, z_0)$ is a point on S. Let C_1 and C_2 be the curves obtained by intersecting the vertical planes $y = y_0$ and $x = x_0$ with the surface S. Then the point P lies on both C_1 and C_2. Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at the point P. Then the *tangent plane* to the surface S at the point P is defined to be the plane that contains both tangent lines T_1 and T_2.

Note. The tangent plane at P is the plane that most closely approximates the surface S near the point P.

Equation of a tangent plane.

Example 1. Find the tangent plane to the elliptic paraboloid $z = 3x^2 + 2y^2$ at the point $(1, 1, 5)$.

Solution.

Linear approximations. Since the tangent plane at a point is a good approximation to the surface near that point, a good approximation to the function $f(x, y)$ (in Example 1) near the point $(1, 1)$ is

$$L(x, y) = 6x + 4y + 5$$

The function L is called the *linearization of* f at $(1, 1)$ and the approximation

$$f(x, y) \approx 6x + 4y + 5$$

is called the *linear approximation* or *tangent plane approximation of* f at $(1, 1)$.

Definition 2. The *linearlization* of $f(x, y)$ at (a, b) is

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

and the *linear approximation* of f at (a, b) is $f(x, y) \approx L(x, y)$.

Definition 3. Let $z = f(x, y)$. Suppose x changes from a to $a + \Delta x$ and y changes from b to $b + \Delta y$. The corresponding *increment* of z is

$$\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$$

and we say that f is *differentiable* at (a, b) if Δz can be written

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

Theorem 1. *If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).*

Example 2. Show that $f(x, y) = xe^{xy}$ is differentiable at $(1, 0)$ and find its linearization there. Then use it to approximate $f(1.1, -0.1)$.

Solution.

Differentials. Recall that if $y = f(x)$, the differential dy is $dy = f'(x)\,dx$, and represents the change in height of the tangent line as x changes by an amount $dx = \Delta x$.
When considering a function of two variables $f(x, y)$, we let dx and dy be independent variables (they are given values).

Definition 4. The *differential* dz, also known as the *total differential*, is defined by

$$dz = f_x(x, y)dx + f_y(x, y)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

Note that the differential dz represents the change in height of the tangent plane, whereas Δz represents the change in height of the surface $z = f(x, y)$.

Example 3. (a) If $z = f(x, y) = x^2 + 3xy - y^2$, find the differential dz.
(b) If x changes from 2 to 2.05 and y changes from 3 to 2.96, compare the values of Δz and dz.

Solution.

Example 4. The base radius and height of a right circular cone are measured as 10 cm and 25 cm, respectively, with a possible error in measurement of as much as 0.1 cm in each. Use differentials to estimate the maximum error in the calculated volume of the cone.

Solution.