14.5 – The chain rule
University of Massachusetts Amherst
Math 233 – Fall 2013

Recall from Calculus I that if $y = f(x)$ and $x = g(t)$, where $f$ and $y$ are differentiable functions, then the chain rule says that

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

Now let $z = f(x, y)$ be a function of $x$ and $y$, which are in turn functions of $t$, say $x = g(t)$ and $y = h(t)$.

**Theorem 1** (Chain rule)**.** *Suppose $z = f(x, y)$ is a differentiable function of $x$ and $y$, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of $t$. Then $z$ is a differentiable function of $t$ and*

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

*Proof.* Use the definition of differentiability from Section 14.4:

**Example 1.** If $z = 2x^2y + 3x^2y^3$, where $x = \cos 2t$ and $y = \sin t$, find $dz/dt$ when $t = 0$.

**Solution.**

**Example 2.** The pressure $P$ (in kilopascals), volume $V$ (in liters), and temperature $T$ (in kelvin) of a mole of an ideal gas are related by the equation $PV = 8.31T$. Find the rate at which the pressure is changing when the temperature is 300 K and increasing at a rate of 0.1 K/s and the volume is 100 L and increasing at a rate of 0.2 L/s.

**Solution.**

**Theorem 2** (Chain rule)**.** *Suppose that $z = f(x, y)$ is a differentiable function of $x$ and $y$, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of $s$ and $t$. Then*

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \textit{and} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

**Example 3.** If $z = e^y \sin x$, where $x = st^2$ and $y = s^2 t$, find $\partial z/\partial s$ and $\partial z/\partial t$.

**Solution.**

**Theorem 3** (Chain rule (general version)). *Suppose that $u$ is a differentiable function of the $n$ variables $x_1, x_2, \ldots, x_n$ and each $x_j$ is a differentiable function of the $m$ variables $t_1, t_2, \ldots, m$. Then $u$ is a function of $t_1, t_2, \ldots, t_m$ and*

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

**Example 4.** If $u = x^4y + y^2z^3$, where $x = rse^t$, $y = rs^2e^{-t}$, and $z = r^2s\sin t$, find the value of

(a) $\partial u/\partial s$ when $r = 2$, $s = 1$, and $t = 0$.
(b) $\partial u/\partial t$ when $r = 2$, $s = 1$, and $t = 0$.

**Solution.**