14.7 – Maxima and minima
University of Massachusetts Amherst
Math 233 – Fall 2013

Definition 1. A function of two variables has a *local maximum* at (a, b) if $f(x, y) \leq f(a, b)$ when (x, y) is near (a, b), and $f(a, b)$ is a *local maximum value*.

Similarly, f has a *local minimum* at (a, b) if $f(x, y) \geq f(a, b)$ when (x, y) is near (a, b), and $f(a, b)$ is a *local minimum value*.

Theorem 1. *If $f(x, y)$ has a a local maximum or local minimum at (a, b) and f_x and f_y exist, then $f_x(a, b) = f_y(a, b) = 0$.*

Definition 2. A point (a, b) is a *critical point* of $f(x, y)$ if $f_x(a, b) = f_y(a, b) = 0$, or if one of these partial derivatives does not exist.

Theorem 2 (Second derivative test)**.** *Suppose the second partial derivatives of $f(x, y)$ are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = f_y(a, b) = 0$, i.e. (a, b) is a critical point of f. Let*

$$D = D(a, b) = \begin{vmatrix} f_{xx}(a, b) & f_{xy}(a, b) \\ f_{yx}(a, b) & f_{yy}(a, b) \end{vmatrix} = f_{xx}(a, b) f_{yy}(a, b) - f_{xy}(a, b)^2.$$

(1) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
(2) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
(3) If $D < 0$, then $f(a, b)$ is a saddle point*. (It is nether a maximum nor a minimum.)*
(4) If $D = 0$, this test is inconclusive.

Example 1. Identify the critical points of
(a) $f(x, y) = x^2 + y^2 - 2x - 6y + 14$,
(b) $f(x, y) = x^4 + y^4 - 4xy + 1$.

Example 2. A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box.

Theorem 3 (Extreme value theorem)**.** *If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.*

Note. The extremal points of f in a bounded region D will be at the critical points of f or on the boundary of D.

Example 3. Find the absolute maximum and minimum values of the function $f(x, y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x, y) \colon 0 \leq x \leq 3, 0 \leq y \leq 2\}$.