14.8 – Lagrange multipliers
University of Massachusetts Amherst
Math 233 – Fall 2013

In this section we develop a method to maximize or minimize a function $f(x, y, z)$ subject to one or more constraints of the form $g(x, y, z) = k$. One way to motivate Lagrange multipliers is to look at the level curves of a function $f(x, y)$ and the curve of a constraint, $g(x, y) = k$.

Geometric interpretation of Lagrange multipliers.

Method of Lagrange multipliers. To find the maximum and niminum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ [assuming that these extreme values exist and $\nabla g \neq \vec{0}$ on the surface $g(x, y, z) = k$]:

(a) Find all values of x, y, z, and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

and

$$g(x, y, z) = k$$

(b) Evaluate f at all the points (x, y, z) that result from step (a). The largest of these values is the maximum value of f; the smallest is the minimum value of f.

Note. We can alternately write the equations in step (a) above as

$$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad f_z = \lambda g_z \qquad g(x, y, z) = k$$

which is a system of four equation in the four unknowns x, y, z, and λ.

For function of two variables, to find the extreme values of $f(x, y)$ subject to $g(x, y) = k$, we look for x, y, and λ such that

$$\nabla f(x, y) = \lambda \nabla g(x, y) \qquad \text{and} \qquad g(x, y) = k$$

or

$$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad g(x, y) = k$$

Example 1. Find the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

Solution.

Example 2. Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are closest to and farthest from the point $(3, 1, -1)$.

Solution.

Two constraints. Suppose we want to maximize $f(x, y, z)$ subject to the constraints $g(x, y, z) = k$ and $h(x, y, z) = c$. Geometrically, this means we are looking for extreme values of f when (x, y, z) is restricted to lie on the curve of intersection C of the level surfaces $g(x, y, z) = k$ and $h(x, y, z) = c$. If (x_0, y_0, z_0) is such an extreme point then there exist Lagrange multipliers λ and μ such that

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0)$$

In other words, to find extreme values we must solve the five equations

$$f_x = \lambda g_x + \mu h_x \qquad f_y = \lambda g_y + \mu h_y \qquad f_z = \lambda g_z + \mu h_z \qquad g(x, y, z) = k \qquad h(x, y, z) = c$$

for the unknowns x, y, z, λ, and μ.

Example 3. Find the maximum value of the function $f(x, y, z) = x + 2y + 3z$ on the curve of intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$.

Solution.