# 15.3 – Double Integrals over General Regions
## University of Massachusetts Amherst
## Math 233 – Fall 2013

**Theorem 1** (Type I region)**.** *If $f(x, y)$ is continuous on a region $D$ such that*

$$D = \{(x, y) \colon a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\},$$

*then*

$$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

**Theorem 2** (Type II region)**.** *If $f(x, y)$ is continuous on a region $D$ such that*

$$D = \{(x, y) \colon h_1(y) \leq x \leq h_2(y), c \leq y \leq d\},$$

*then*

$$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x, y)\, dx\, dy.$$

**Example 1.** Evaluate $\iint_D x + 2y\, dA$, where $D$ is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

**Example 2.** Find the volume of the solid hat lies under the paraboloid $z = x^2 + y^2$ and above the region $D$ in the $xy$-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

**Example 3.** Evaluate $\iint\limits_D xy\,dA$, where $D$ is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

**Example 4.** Evaluate the iterated integral $\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx$.

**Properties of double integrals:** Let $f(x,y)$ and $g(x,y)$ be integral functions over a region $D$, and let $c$ be a constant.

(i) $\iint\limits_D f(x,y) + g(x,y)\, dA = \iint\limits_D f(x,y)\, dA + \iint\limits_D g(x,y)\, dA.$

(ii) $\iint\limits_D cf(x,y)\, dA = c \iint\limits_D f(x,y)\, dA.$

(iii) $\iint\limits_D f(x,y)\, dA = \iint\limits_{D_1} f(x,y)\, dA + \iint\limits_{D_2} f(x,y)\, dA,$

if $D = D_1 \cup D_2$, and $D_1$ and $D_2$ are disjoint regions.

(iv) $\iint\limits_D 1\, dA = A(D)$, the area of $D$.

(v) If $m \leq f(x,y) \leq M$ for all $(x,y) \in D$, then

$$mA(D) \leq \iint\limits_D f(x,y)\, dA \leq MA(D).$$

**Example 5.** Find bounds for the integral $\iint_D e^{\sin(x)\cos(y)}\, dA$, where $D$ is the disk of radius 2 centered at the origin.