15.4 – Double Integrals in Polar Coordinates
University of Massachusetts Amherst
Math 233 – Fall 2013

It is often easier to compute a double integral over certain regions, such as a circle or annulus, using polar coordinates. Recall that polar coordinates (r, θ) are defined in terms of rectangular coordinates (x, y) by

$$r^2 = x^2 + y^2 \qquad x = r\cos\theta \qquad y = r\sin\theta$$

Definition 1. A *polar rectangle* has the form

$$R = \{(r, \theta) \mid a \le r \le b, \alpha \le \theta \le \beta\}$$

and can be used to describe full circles or annuli, or portions of those shapes.

Polar coordinates - graphically.

If R is a polar rectangle, how do we compute $\iint_R f(x, y)\, dA$? If we divide the rectangle into subrectangles $[r_{i-1}, r_i] \times [\theta_{j-1}, \theta_j]$, note that each has "center"

$$r_i^* = \frac{1}{2}(r_{i-1} + r_i) \qquad \theta_j^* = \frac{1}{2}(\theta_{j-1}, \theta_j)$$

Furthermore, since the area of a sector of a circle with radius r and central angle θ is $r^2\theta/2$, then if $\Delta\theta = \theta_j - \theta_{j-1}$, the area of one of the subrectangles is

$$\Delta A_i = \frac{1}{2}r_i^2\Delta\theta - \frac{1}{2}r_{i-1}^2\Delta\theta = \frac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})\Delta\theta = r_i^*\Delta r\Delta\theta$$

We then have

$$\begin{aligned}
\iint_R f(x, y)\, dA &= \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*)\Delta A_i \\
&= \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*) r_i^*\Delta r\Delta\theta \\
&= \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta
\end{aligned}$$

Change to Polar Coordinates in a Double Integral. If f is continuous on a polar rectangle R given by $0 \le a \le r \le b$, $\alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$, then

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Important! Don't forget that r!

Example 1. Evaluate $\iint_R (3x + 4y^2)\,dA$, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Example 2. Find the volume of a sphere of radius a.

Example 3. Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.